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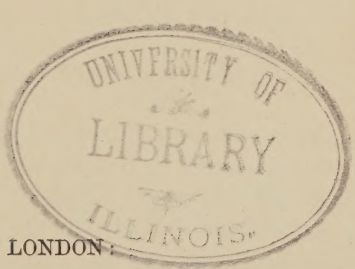
PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XV.

FROM NOVEMBER, 1883, TO NOVEMBER, 1884.



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PROCEEDINGS

OF THE

LONDON MATHEMATICAL SOCIETY.

VOL. XV.

TWENTIETH SESSION, 1883–4.

November 8th, 1883.

ANNUAL GENERAL MEETING, held at 22 Albemarle Street, W.

Professor HENRICI, F.R.S., President, in the Chair.

The President and Dr. Hirst spoke upon the loss the Society had sustained by the death of Dr. Spottiswoode, P.R.S., and the following resolution was carried unanimously: — "That the Secretaries be requested to communicate to Mrs. Spottiswoode, the expression of our sincere sympathy, and the assurance of our deep sense of the loss which Science has sustained by the untimely death of Mr. Spottiswoode."

The Treasurer (Mr. A. B. Kempe) read his report. Its reception was moved by Mr. R. F. Scott, seconded by Mr. W. J. C. Sharp, and carried unanimously.

At the request of the Chairman, Professor Greenhill consented to act as Auditor.

From the report of the Secretaries, it appeared that the number of members since the last General Meeting, held November 9th, 1882, had increased from 165 to 169, of these 57 being Life Members.

The obituary of the Society comprised the following names :—

Dr. W. Spottiswoode, P.R.S., elected June 19th, 1865—died June 27th, 1883.

Professor H. J. S. Smith, F.R.S., elected October 16th, 1865—died February 9th, 1883.

Charles Robert Rippin, M.A., elected October 16th, 1865—died February 23rd, 1883.

Dr. W. Robertson, elected April 25th, 1867—died August 25th, 1882.

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Cecil James Monro, M.A., elected May 11th, 1871—died November 25th, 1882.
Enoch B. Seitz, M.A., elected March 11th, 1880—died October 8th, 1883.

The following communications had been made:

Presidential Address by S. Roberts, F.R.S., entitled "Remarks on Mathematical Terminology and the Philosophical Bearing of recent Mathematical Speculations concerning the Realities of Space."

"On In- and Circumscribed Polyhedra:" Professor Forsyth, M.A.

"Note on Quartic Curves in Space:" Dr. Spottiswoode, P.R.S.

"On Unicursal Twisted Quartics, Parts I., II.:" R. A. Roberts, M.A.

"Note on the Derivation of Elliptic Function Formulæ from Confocal Conics:" J. Griffiths, M.A.

"On the Explicit Integration of certain Differential Resolvents:" Sir J. Cockle, F.R.S.

"On Compound Determinants:" R. F. Scott, M.A.

"On the Vibrations of a Spherical Shell:" Professor Lamb, M.A.

"On the absolutely Least Periods of the Elliptic Functions:" Professor H. J. S. Smith, F.R.S.

"On certain Relations between Volumes of Loci of connected Points:" E. B. Elliott, M.A.

"Geometrical Proof of Griffiths's Extension of Graves's Theorem:" J. J. Walker, M.A.

"On Polygons circumscribed about a Tricuspidal Quartic:" R. A. Roberts, M.A.

"Note on an exceptional case in which the Fundamental Postulate of Professor Sylvester's Theory of Tamisage fails:" J. Hammond, M.A.

"On certain Quartic Curves which have a Cusp at Infinity, whereat the Line at Infinity is a Tangent:" H. M. Jeffery, F.R.S.

"On the Automorphic Transformation of a Binary Cubic Function:" Professor Cayley, F.R.S.

"On the Sylvester-Kempe Quadruplane:" H. Hart, M.A.

"On Curves obtained by an extension of Maclaurin's Method of constructing Conics:" S. Roberts, F.R.S.

"A Generalisation of the Nine Points Properties of a Triangle:" Captain P. A. MacMahon, R.A.

"On the Use of certain Differential Operators in the Theory of Equations:" J. Hammond, M.A.

"A Method for reducing the Differential Expression $dt / \sqrt{\{t-\alpha \,.\, t-\beta \,.\, t-\gamma \,.\, t-\delta\}}$ to the Standard Form:" J. Griffiths, M.A.

"Calculation of the Hyperbolic Logarithm of π:" J. W. L. Glaisher, F.R.S.

"On Monge's 'Mémoire sur la Théorie des Déblais et des Remblais':" Professor Cayley, F.R.S.

"Equations of the Loci of the Intersections of Three Tangent Lines and of Three Tangent Planes to any Quadric $u = 0$:" Professor Wolstenholme, M.A., Sc.D.

"Investigation of the character of the Equilibrium of an Incompressible Heavy Fluid of Variable Density:" Lord Rayleigh, F.R.S.

"On the Normal Integrals connected with Abel's Theorem:" Professor Forsyth, M.A.

"On Elliptic Function Formulæ connected with the Transformation of Rectangular Coordinates:" Rev. M. M. U. Wilkinson, M.A.

"Calculation of the Equation which determines the Anharmonic Ratios of the Roots of a Quintic:" Professor M. J. M. Hill, M.A.

"On Simultaneous Differential Equations, with special reference to (1) the Roots

of the Fundamental Determinant, (2) the Method of Multipliers :" E. J. Routh, F.R.S.

"On Mr. Wilkinson's Rectangular Transformation :" Professor Cayley, F.R.S.

"Relations between the Common Points and Common Tangents of Two Conics :" Professor R. W. Genese, M.A.

"On two Concentric Circles :" R. Tucker, M.A.

"On the Motion of a Particle on the Surface of an Ellipsoid :" W. R. W. Roberts, M.A.

"On the Mutual Potential of Two Lines in Space :" Professor H. Lamb, M.A.

"On Bicircular Quartics with Collinear Foci :" H. M. Jeffery, F.R.S.

Minor communications were made by Dr. Hirst, F.R.S., Messrs. J. W. L. Glaisher, F.R.S., J. Hammond, M.A., and R. Tucker, M.A.

Additional exchanges of *Proceedings* have been made with Prof. Teixeira ("Jornal de Sciencias Mathematicas," Coimbra); Prof. G. Mittag-Leffler ("Acta Mathematica"); and the "Reale Accademia della Scienze di Napoli."

Messrs. Sharp and Gerrans having consented to act as Scrutators, the meeting proceeded to the election of the new Council. On an examination of the balloting papers, the following gentlemen were declared duly elected :—

President, Dr. O. Henrici, F.R.S.; Vice-Presidents, Sir J. Cockle, Knt., F.R.S., Samuel Roberts, Esq., F.R.S.; Treasurer, A. B. Kempe, Esq., F.R.S.; Secretaries, M. Jenkins, Esq., M.A., R. Tucker, Esq., M.A.; Other Members of the Council—Professor Cayley, F.R.S., E. B. Elliott, Esq., M.A., J. W. L. Glaisher, Esq., F.R.S., J. Hammond, Esq., M.A., H. Hart, Esq., M.A., Dr. Hirst, Esq., F.R.S., W. D. Niven, Esq., F.R.S., Professor Rowe, M.A., R. F. Scott, Esq., M.A., J. J. Walker, Esq., F.R.S.

The President thanked the members of the Society for his re-election. The Rev. J. J. Milne, M.A., late Scholar of St. John's College, Cambridge, and Mr. F. W. Watkin, B.A., formerly Scholar of Corpus Christi College, Oxford, were elected members.

The following papers were read, or communicated in part, viz. :—

"Symmetric Functions, and in particular on Certain Inverse Operators in connection therewith :" Captain P. A. MacMahon, R.A. "On a Certain Envelope :" Professor Wolstenholme. "On Certain Results obtained by means of the Arguments of Points on a Plane Curve :" R. A. Roberts, M.A. Third Paper on "Multiple Frullanian Integrals :" E. B. Elliott, M.A. "Note on Jacobi's Transformation of Elliptic Functions :" J. Griffiths, M.A. "Symmedians and the Triplicate-Ratio Circle :" R. Tucker, M.A.

The following presents were received :—

"Educational Times" for November, 1883.

"Nautical Almanac" for 1887.

"Great Trigonometrical Survey of India," Vol. ix., 1883.

"Smithsonian Report," 1881.

"Beiblätter zu den Annalen der Physik und Chemie," Band vii., Stück 10.

"Journal für die reine und angewandte Mathematik"—(Crelle), Bd. xcv., Heft 2.
"Acta Mathematica," i., 4; ii., 1; ii., 2.
"Atti della R. Accademia dei Lincei," Vol. 7, Fasc. 15.
"Journal de l'Ecole Polytechnique," 53d cahier; Paris, 1883.
Cabinet likenesses of Rev. R. Harley, F.R.S., and Rev. Dr. Logan.

On Certain Results obtained by means of the Arguments of Points on a Plane Curve. By R. A. Roberts, M.A.

[*Read Nov. 8th*, 1883.]

1. We know that there are certain relations connecting the arguments of collinear points on a plane curve, the argument of a point on the curve being

$$\int \frac{dx}{\frac{dF}{dy}} \quad \text{or} \quad \int \frac{ds}{\sqrt{\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2\right\}}},$$

where $F(x, y) = 0$ is the equation of the curve in rectangular Cartesian coordinates. But I am not aware whether it has been noticed that, if the line containing the points touch an envelope, the ratios of the differentials of the arguments can be expressed in terms of the distances of the points on the curve from the point of contact with the envelope. By means of this result several geometrical theorems can be arrived at with respect to triangles, or figures of more sides, inscribed in one curve and circumscribed about another.

2. For the cubic, the ratios of the differentials of the arguments of three collinear points on the curve are the anharmonic ratios of the three points and the point where the line touches its envelope. I give a proof of this theorem which can be applied to give similar results for curves of higher degree.

Suppose the cubic, a point A on the curve being origin, to be written in the form

$$U \equiv v_3+v_2+v_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where V_n denotes the terms of the n^{th} degree in x and y; then, transforming to polar coordinates, we see that, if a line which makes an angle ϑ with the axes of x meet the curve again in B, C, we must have

$$AB \,.\, AC = \frac{g\cos\vartheta + f\sin\vartheta}{M} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where we have put $v_1 = gx+fy$, and M is a function depending only on the direction of the line ABC. But $gx+fy = 0$ is the tangent to

the curve at A; hence, if ϕ is the angle which ABC makes with this line, we have $g\cos\vartheta + f\sin\vartheta = \sqrt{(g^2+f^2)}\sin\phi$. Now g^2+f^2 is evidently the value of $\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2$ at the origin, and this function, as is well known, is independent of any system of rectangular axes to which the curve is referred. Thus (2) gives

$$AB\,.\,AC = \frac{\sin\phi}{M}\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Hence, if ϕ_1, ϕ_2, ϕ_3 are the angles which the line makes with the curve at A, B, C, respectively, and P_1, P_2, P_3 are the values of

$$\sqrt{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2\right\}}$$

corresponding to the same points, we have, from (3),

$$AB\,.\,AC = \frac{P_1\sin\phi_1}{M},\quad BA\,.\,BC = \frac{P_2\sin\phi_2}{M},\quad CA\,.\,CB = \frac{P_3\sin\phi_3}{M} \ldots\ldots(4).$$

Now, if the line ABC touch its envelope at O, it is easy to see, by infinitesimals, that we have

$$OA\,d\omega = \sin\phi_1\,ds_1,\quad OB\,d\omega = \sin\phi_2\,ds_2,\quad OC\,d\omega = \sin\phi_3\,ds_3 \ldots\ldots(5),$$

where ds_1, ds_2, ds_3 are the elements of the arcs of the curve at A, B, C, respectively, and $d\omega$ is the angle between two consecutive positions of the line.

Hence, if we put $ds_1 = P_1\,du_1$, $ds_2 = P_2\,du_2$, $ds_3 = P_3\,du_3$, we obtain, from (4) and (5),

$$\frac{du_1}{OA\,.\,BC} = \frac{du_2}{OB\,.\,CA} = \frac{du_3}{OC\,.\,AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

which gives the result I have stated above.

For curves of the fourth order, we find, in a similar manner,

$$du_1 : du_2 : du_3 : du_4$$

$$= \frac{OA}{AB\,.\,AC\,.\,AD} : \frac{OB}{BA\,.\,BC\,.\,BD} : \frac{OC}{CA\,.\,CB\,.\,CD} : \frac{OD}{DA\,.\,DB\,.\,DC} \ldots(7)$$

and similar expressions for curves of higher degrees.

3. I proceed to deduce some results from (6). If we take two corresponding points A, B on a cubic, we have

$$u_1 - u_2 = \tfrac{1}{2}\omega \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where $\omega \equiv 2mK + 2m'K'$; for, since the tangents at these points intersect at a point u on the curve, we get, from the relation connecting three collinear points, $2u_1 + u = \alpha$, a constant, and $2u_2 + u = \alpha$; hence we get (8). We have, therefore, $du_1 = du_2$, and, from (6), $OA\,.\,BC = OB\,.\,CA$. We thus obtain the known theorem that the

tangent to the Cayleyan is divided harmonically at the pair of corresponding points on the Hessian, the point of contact, and the point where it meets the Hessian again.

4. Suppose a triangle inscribed in a cubic so that the tangents at the vertices pass through the points where the opposite sides meet the curve again; then, if u_1, u_2, u_3 are the arguments of the vertices, and v_1, v_2, v_3 those of the points where the sides meet the curve again, we have $2u_1+v_1=a$, and $u_2+u_3+v_1=a$; therefore $2u_1-u_2-u_3=0$, or ω, and similarly $2u_2-u_1-u_3$ and $2u_3-u_1-u_2$ are respectively equal to a multiple of ω. Hence we may take $u_2-u_3=\frac{1}{3}\omega$, $u_3-u_1=\frac{1}{3}\omega$, $u_1-u_2=-\frac{2}{3}\omega$.

Thus we see that these three conditions are not independent, but are only equivalent to two. Hence there are an infinite number of such triangles, and, since from $u_1-u_2=\frac{1}{3}\omega$ we have $du_1=du_2$, a side of one of these triangles is divided harmonically at its point of contact and the three points where it meets the curve.

5. If we take pairs of points P, P' on a cubic so that P, Q are collinear with a fixed point A, and P', Q with a fixed point B, the points A and B being on the curve, then we have $u_1+v+a=a$, $u_2+v+b=a$, and therefore $u_1-u_2=b-a$, a constant. Hence, as before, we see that the line PP' is divided harmonically at P, P', the point of contact with its envelope, and the point where it meets the curve again. We can show from this that the envelope, which is of the sixth class, touches the given curve eighteen times. This result includes the two preceding as particular cases.

6. Suppose a curve of the m^{th} degree to have p-pointic contact with a cubic at the point u_1 and to meet the cubic again at a variable point u_2 and $3m-p-1$ fixed points, then, by Clebsch's theory of the arguments of points on a cubic, we have $pu_1+u_2=$ a constant.

Hence we see, from (6), that the line joining the points u_1, u_2 is divided in a constant anharmonic ratio at these points, the point of contact with its envelope, and the point where it meets the curve again. It may be shown that the class of the envelope is $2(p^2-p+1)$.

7. In a paper in No. 189, Vol. xiii., p. 150, of the *Proceedings*, I have remarked that, if we take collinear points on the cuspidal cubic $U\equiv y^3-x^2z=0$, whose parameters are subject to the relations

$$\Sigma\vartheta_1=0,\quad (\Sigma\vartheta_3\vartheta_2)^2+27k(\vartheta_1\vartheta_2\vartheta_3)^2=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

then the tangents at these points form a triangle inscribed in the cubic $V\equiv y^3-kx^2z=0$. Since from (9) we see that the parameters are in a constant ratio to one another, it follows that the line which touches $ky^3-x^2z=0$ is divided in a constant anharmonic ratio at the three points on U and the point of contact. Also, by recipro-

cation, the tangents to V at the vertices pass through a point on the cubic $y^3-k^2x^2z$, and these tangents and the tangent to the latter cubic have a constant anharmonic ratio.

8. If we put $x=\vartheta y$ in the equation of the cuspidal cubic $U\equiv y^3-x^2z=0$, the coordinates of the intersection of the tangents ϑ_1, ϑ_2 are

$$x=\tfrac{1}{2}\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2),\quad y=\tfrac{1}{3}(\vartheta_1^2+\vartheta_2^2+\vartheta_1\vartheta_2),\quad z=1.$$

Hence if $\vartheta_1-\vartheta_2=c$, a constant (9), we have

$$x=\tfrac{1}{2}\vartheta_1\vartheta_2(\vartheta_1+\vartheta_2),\quad cy=\tfrac{1}{3}(\vartheta_1^3-\vartheta_2^3),\quad c^3z=(\vartheta_1-\vartheta_2)^3;$$

whence, eliminating $\dfrac{\vartheta_1}{\vartheta_2}$, we obtain

$$(y+kz)(y+4kz)^2-x^2z=0 \quad \text{.....(10)},$$

where $k=-12c^2$, which represents a nodal cubic. But from $d\vartheta_1=d\vartheta_2$ we see that the tangent to (10) is a fourth harmonic to the three tangents drawn to U. Again, it is evident that the tangents to the three curves of the system (10) which pass through a point P are the three fourth harmonics to the tangents drawn from P to U.

9. Suppose a variable triangle A, B, C, inscribed in a cubic; and let O_1, O_2, O_3 be the points of contact of the sides with their envelopes, and a, b, c the points in which the sides meet the curve again. Then, if u_1, u_2, u_3 are the arguments of A, B, C, we have, from (6),

$$\frac{du_1}{du_2}=-\frac{AO_3\,.\,Bc}{BO_3\,.\,Ac},\qquad \frac{du_2}{du_3}=-\frac{BO_1\,.\,Ca}{CO_1\,.\,Ba},$$

$$\frac{du_3}{du_1}=-\frac{CO_2\,.\,Ab}{AO_2\,.\,Cb};$$

therefore

$$\frac{AO_3\,.\,BO_1\,.\,CO_2}{BO_3\,.\,CO_1\,.\,AO_2}+\frac{Bc\,.\,Ca\,.\,Ab}{Ac\,.\,Ba\,.\,Cb}=0 \quad \text{.....(11)}.$$

From this result it follows at once, by a known geometrical theorem, that if the lines Aa, Bb, Cc pass through a point, then the points O_1, O_2, O_3 will lie on a line, and *vice versâ*. Now, if the lines Aa, Bb, Cc pass through a point, it can be shown that the tangents to the curve at A, B, C also pass through a point. For, referring the cubic to the triangle ABC, it may be written in the form

$$x(b_1y^2+c_1z^2)+y(a_2x^2+c_2z^2)+z(a_3x^2+b_3y^2)+2mxyz=0.$$

The lines joining the vertices to the points where the opposite sides meet the curve again are then

$$b_1y+a_2x=0,\quad c_1z+a_3x=0,\quad c_2z+b_3y=0 \quad \text{.....(12)},$$

and the tangents at the vertices are

$$c_1x+c_2y=0,\quad b_1x+b_3z=0,\quad a_2y+a_3z=0 \quad \text{.....(13)}.$$

But if the lines (12) pass through a point, we have

$$a_2c_1b_3+a_3b_1c_2=0;$$

which is also the condition that the lines (12) should pass through a point. Now, if tangents be drawn to the cubic from a point P lying on a given locus, the lines joining their points of contact will have a definite envelope, and, from what we have shown above, if the curve is non-singular, these fifteen lines will touch their envelope in points lying by threes on twenty right lines.

10. In the paper referred to above, I have shown that an infinite number of triangles can be circumscribed about the cuspidal cubic $U\equiv y^3-x^2z=0$, so as to be inscribed in a cuspidal cubic V, this curve being connected with U by certain relations. Now, if two further relations are satisfied, the points of contact of the sides of the triangle will always lie on a line; in fact, in this case, the parameters ϑ_1, ϑ_2, ϑ_3 are connected by the equations

$$\vartheta_1+\vartheta_2+\vartheta_3=0, \quad a(\vartheta_1\vartheta_2+\vartheta_2\vartheta_3+\vartheta_3\vartheta_1)+b\vartheta_1\vartheta_2\vartheta_3=0,$$

where a and b are constants. Hence, from what we have shown above, we see that in this case the tangents to V at the vertices of the triangle always pass through a point.

11. Suppose the points a, b, c to lie on a line, then from (11) the lines AO_1, BO_2, CO_3 pass through a point. Hence we see that, if a series of quadrilaterals A, B, C, a, b, c have their six intersections of sides on a cubic, and if the points of contact of two of the sides with their envelopes be given, then the points of contact of the remaining two sides can be constructed at once.

12. If it be possible to inscribe a series of triangles in a cubic which shall be circumscribed about a fixed conic, we know that the lines AO_1, BO_2, CO_3 pass through a point, and, therefore, as we have seen, the points a, b, c lie on a line. But, when a quadrilateral has its six intersections of sides on a cubic, the extremities of the diagonals are corresponding points on the curve, and therefore from (8) the differentials of the arguments of these points are equal. Hence, from (6), we see that the anharmonic ratio of the three points on a side and the point of contact of that side with its envelope has the same value for each of the four sides of the quadrilateral. But this is evidently the result which we should obtain by supposing a line divided in a constant anharmonic ratio by four fixed lines, to coincide with each of the fixed lines in turn. Thus we see that the fourth side abc must touch the conic. Now this result coincides with what we have known before—namely, that there exist conics about which an infinite number of quadrilaterals can be circumscribed so as to have their six inter-

sections of sides on a cubic; but, by the method which I have employed, it appears that these conics are the *only* system about which an infinite number of triangles can be circumscribed so as to be inscribed in the cubic.

13. Let us suppose a triangle A, B, C inscribed in a curve of the fourth order, and let a, b, c, a', b', c' be the points where the sides meet the curve again, and O_1, O_2, O_3 the points of contact of the sides with their envelopes, then, from (7), we find, as at (11),

$$\frac{AO_3 . BO_1 . CO_2}{BO_3 . CO_1 . AO_2} - \frac{Bc . Bc' . Ca . Ca' . Ab . Ab'}{Ac . Ac' . Ba . Ba' . Cb . Cb'} = 0 \quad \ldots\ldots\ldots(14).$$

Hence we see, by Carnot's theorem, that if the lines AO_1, BO_2, CO_3 pass through a point, then the points a, b, c, a', b', c' will lie on a conic. This will be the case if the triangle is always circumscribed about a fixed conic. I am not aware whether such a conic exists for the general quartic, but it can be found for certain special quartics. If the quartic has a double point, such a conic exists; for in the mode of generation of a general quartic as the envelope of a series of conics having a common Jacobian (Salmon's *Higher Plane Curves*, Art. 251), if we suppose the conics to have a common point, this point will evidently be a node of the quartic. Now the Cayleyan is the envelope of the lines joining the points of contact of the variable system of conics (*Curves*, Art. 253), and, in this case, it breaks up into a point and a conic. We thus have a series of triangles inscribed in the quartic and circumscribed about a fixed conic. We see then, from what we have shown above, that the points where the sides of one of these triangles meet the curve again lie on a conic.

14. We can verify the result (14) in a certain case. If the invariant B of a quartic vanish, it can be expressed linearly in an infinite number of ways in terms of the fourth powers of five lines. These lines all touch a fixed conic; namely, a conic such that, if we substitute differential symbols in its tangential equation and operate on the quartic, the result will vanish. If we take the covariant S (Salmon's *Higher Plane Curves*, Art. 297) of this quartic, we get another quartic which passes through all the intersections of the five lines. We thus have an infinite number of triangles circumscribed about a conic, and inscribed in the latter quartic. In this case, then, the conic passing through the points where the sides meet the curve again breaks up into two lines.

15. We can find, by the method employed above, for triangles inscribed in a curve of the n^{th} degree, a relation connecting the points of contact of the sides with their envelopes and the points where the sides meet the curve again. As a particular case, we find that, if it be

possible to circumscribe about a conic a series of triangles which shall be inscribed in a curve of the n^{th} degree, then the $3(n-2)$ points where the sides meet the curve again will lie on a curve of the $(n-2)^{\text{th}}$ degree. This result may, however, be obtained directly by means of Carnot's theorem. For, since two triangles circumscribed about a conic have their vertices on another conic, it follows that, if we consider two consecutive triangles, the vertices must be the points of contact of a conic touching the curve three times. But, if we make use of Carnot's theorem for a triangle inscribed in a curve of the n^{th} degree, we obtain a relation connecting the intercepts on the sides and the sines of the angles which the sides make with the tangents to the curve at the vertices (Salmon's *Higher Plane Curves*, Art. 126). Now the factors containing the sines of the angles will disappear in consequence of the fact that the tangents to the curve at the vertices are tangents to a conic passing through these points, and the remaining relation connecting the segments expresses the result I have stated above.

16. If an infinite number of polygons of m sides are inscribed in a curve of the n^{th} degree, we can find a relation connecting the points where the sides touch their envelopes and the points where they meet the curve again.

This relation is, in fact,

$$\frac{AO_1 \,.\, BO_2 \,.\, \&c.}{AO_m \,.\, BO_1 \,.\, \&c.} \times \frac{(A)'\,(B)'\,\&c.}{'(A)\,'(B)\,\&c.} = (-1)^{mn+1} \ldots\ldots\ldots\ldots (15),$$

where A, B, C, &c. are the vertices of the polygon, O_1, O_2, &c. the points of contact of AB, BC, &c., and $(B)'$, $'(B)$ denote the continued products of the $n-2$ segments made on the sides BC and BA, respectively, between B and the curve. This theorem can be applied to several particular cases, but does not appear to give any geometrical results of interest.

17. By means of the results obtained in § 2, we can write down a system of relations connecting the points on a curve which lie on a line touching a given conic. For, if we take points on a tangent to a conic S, their distances from the point of contact are proportional to the square roots of the results of substituting their coordinates in the equation of S. In the case of the cubic, we have then, from (6),

$$\frac{du_1}{\sqrt{S_1}} : \frac{du_2}{\sqrt{S_2}} : \frac{du_3}{\sqrt{S_3}} = BC : CA : AB;$$

therefore $\quad \Sigma \frac{du}{\sqrt{S}} = 0, \quad \Sigma \frac{Ldu}{\sqrt{S}} = 0,$

where L is an arbitrary line.

Similarly, for the curve of the n^{th} degree, we have the $n-1$ equations

$$\Sigma \frac{du}{\sqrt{S}} = 0, \quad \Sigma U_1 \frac{du}{\sqrt{S}} = 0 \ \ldots\ldots \ \Sigma U_{n-2} \frac{du}{\sqrt{S}} = 0 \ \ldots\ldots\ldots (16),$$

where U_r is an arbitrary curve of the r^{th} degree.

For curves of any deficiency it is evident that the expression under the radical $\sqrt{S}$ cannot be rationalized; but for unicursal curves the expressions $\int \frac{du}{\sqrt{S}}$, &c. will be hyperelliptic integrals: for then S can be expressed as a polynomial of the $(2n)^{\text{th}}$ degree in a parameter. For non-unicursal curves, however, in certain particular cases, some of the expressions (16) will admit of simplication; as, for instance, let us consider a conic S having double contact with a cubic. If a conic S have double contact with a cubic, the cubic can be written in the form $AC^2 - BS = 0$, where A, B, C are lines; hence, for any point on the cubic,

$$\sqrt{S} = C\sqrt{\left(\frac{A}{B}\right)}, \text{ and } \frac{L du}{\sqrt{S}} \text{ becomes } \frac{L}{C}\sqrt{\left(\frac{B}{A}\right)} du = \sqrt{\left(\frac{B}{A}\right)} du,$$

if the line L is taken so as to coincide with C. Now, if we take any point O on the curve as origin, it is easy to see that for any point P on the curve

$$du \propto \frac{dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\delta)\}}},$$

where t is the tangent of the angle which OP makes with a fixed line, and α, β, γ, δ are the values of t corresponding to the four tangents drawn from O to the curve. Now, suppose O to be the point AB, then, since B is one of the tangents from AB, we may put

$$\frac{B}{A} = \frac{t-\delta}{t-\epsilon},$$

where ϵ is the value of t corresponding to the line A.

Hence $$\sqrt{\left(\frac{B}{A}\right)} du = \frac{dt}{\sqrt{\{(t-\alpha)(t-\beta)(t-\gamma)(t-\epsilon)\}}} = d\nu, \text{ say,}$$

where ν is an elliptic integral; thus we have

$$\nu_1 \pm \nu_2 \pm \nu_3 = \text{a constant.}$$

If the conic S have triple contact with the curve, A is a tangent, and we may take then $\epsilon = \gamma$, in which case ν becomes either a logarithm or a circular function.

We can obtain a very simple relation in the case of a conic S touching a bicircular quartic four times. The quartic may then be written in the form $\Sigma^2 - S = 0$, where Σ is a circle; hence

$$\frac{U_2 du}{\sqrt{S}} \text{ becomes } \pm \frac{U_2 du}{\Sigma} = \pm du,$$

if we take U_2 so as to coincide with Σ. Thus we have $\Sigma \pm du = 0$; but we have $\Sigma du = 0$; hence we obtain

$$u_1 + u_2 = \sigma, \text{ a constant,}$$
$$u_3 + u_4 = \omega - \sigma \quad \ldots\ldots\ldots\

Frullani's,
$$\int_0^\infty \left\{ \frac{f'(x)}{x^{r-1}} - \frac{f'(ax)}{(ax)^{r-1}} \right\} \frac{dx}{x} = 0,$$

i.e.,
$$\int_0^\infty f'(ax) \frac{dx}{x^r} = a^{r-1} \int_0^\infty f'(x) \frac{dx}{x^r}.*$$

Hence, integrating between limits with regard to a,
$$\int_0^\infty \{f(ax) - f(bx)\} \frac{dx}{x^{r+1}} = \frac{a^r - b^r}{r} \int_0^\infty f'(x) \frac{dx}{x^r} \quad \ldots\ldots\ldots\ldots (1),$$

the theorem desired. It must never be forgotten that, in proving it, the integral on its right-hand side has been assumed not infinite. Were it so, the equality could have little meaning, and would be absolutely untrue, unless the difference of limits $\left[\frac{f'(x)}{x^{r-1}} \right]_0^\infty$ vanished.

Before proceeding from this to the general theorem, it seems well first to investigate that sub-class of results which is of greatest simplicity and apparent utility, especially as the application to it of (1) is more entirely free from anything unintelligible or unsatisfactory than may possibly seem to be the case with the use of that equality in the derivation of the more comprehensive result; viz.,

I. *The sub-class*
$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \left\{ \prod_{s=1}^{s=n} f(a_s x_s) - \prod_{s=1}^{s=n} f(b_s x_s) \right\} \frac{dx_1 dx_2 \ldots dx_n}{(x_1 x_2 \ldots x_n)^{r+1}}.$$

---

* The remark seems worth making, that in general it is or is not lawful to apply the transformation $x = ay$ to the integral $\int_0^\infty F(x)\,dx$ or $\int_{-\infty}^0 F(x)\,dx$ according as, when x becomes in turn positive infinity and zero, or zero and negative infinity, as the case may be, $xF(x)$ does or does not approach limits whose difference is zero. In particular, it is always lawful when the integral is of finite value, since both limiting values in that case vanish.

In like manner, $x = y + a$ is a lawful substitution to apply to $\int_{-\infty}^\infty F(x)\,dx$, when, and only when, the difference of limits $F(\infty) - F(-\infty)$ vanishes, as is always the case when the value of the integral is finite, since then each limit is equal to zero.

And, once more, according as the difference of limits
$$[x \log x F(x)]_0^1 \text{ or } [x \log x F(x)]_1^\infty$$
vanishes, as will in particular be the case when the value of the integral is finite, or not, so is it allowable or not to transform $\int_0^1 F(x)\,dx$, or $\int_1^\infty F(x)\,dx$, as the case may be, by a substitution like $x = y^a$.

These and other like results are immediate from inspection of Frullani's theorem in the forms
$$\int_0^\infty \{\phi(ax) - \phi(x)\} \frac{dx}{x} = \{\phi(\infty) - \phi(0)\} \log a,$$
$$\int_{-\infty}^\infty \{\phi(x+a) - \phi(x)\}\,dx = \{\phi(\infty) - \phi(-\infty)\}\, a,$$
$$\int_0^1 \{\phi(x^a) - \phi(x)\} \frac{dx}{x \log x} = \{\phi(1) - \phi(0)\} \log a,$$
and others obtained by simple transformations.

The functions, and constants r, dealt with are exactly those for which result (1) has been arrived at.

The double integral of the class,

$$\int_0^\infty \int_0^\infty \{f(a_1x_1)\, f(a_2x_2) - f(b_1x_1)\, f(b_2x_2)\} \frac{dx_1\, dx_2}{(x_1x_2)^{r+1}},$$

may be written as the sum of two double integrals, namely,

$$\int_0^\infty f(a_1x_1) \frac{dx_1}{x_1^{r+1}} \int_0^\infty \{f(a_2x_2) - f(b_2x_2)\} \frac{dx_2}{x_2^{r+1}}$$
$$+ \int_0^\infty f(b_2x_2) \frac{dx_2}{x_2^{r+1}} \int_0^\infty \{f(a_1x_1) - f(b_1x_1)\} \frac{dx_1}{x_1^{r+1}};$$

and this sum is, by (1), equal to

$$\left\{ \frac{a_2^r - b_2^r}{r} \int_0^\infty f(a_1x_1) \frac{dx_1}{x_1^{r+1}} + \frac{a_1^r - b_1^r}{r} \int_0^\infty f(b_2x_2) \frac{dx_2}{x_2^{r+1}} \right\} \int_0^\infty f'(x) \frac{dx}{x^r}.$$

If then we have satisfied the one condition

$$a_2^r - b_2^r + a_1^r - b_1^r = 0,$$

i.e.,
$$a_1^r + a^r = b_1^r + b_2^r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the double integral is equal to

$$-\frac{a_1^r - b_1^r}{r} \int_0^\infty \{f(a_1x) - f(b_2x)\} \frac{dx}{x^{r+1}} \cdot \int_0^\infty f'(x) \frac{dx}{x^r},$$

which, by (1), $= -\dfrac{(a_1^r - b_1^r)(a_1^r - b_2^r)}{r^2} \left\{ \int_0^\infty f'(x) \dfrac{dx}{x^r} \right\}^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$

Now the symmetrical forms of the double integral and of the condition (2) tell us that there are three other forms in which the coefficient of the integral factor in this result might with equal accuracy have been obtained, viz.,

$$-\frac{1}{r^2}(a_2^r - b_1^r)(a_2^r - b_2^r), \qquad +\frac{1}{r^2}(a_1^r - b_1^r)(a_2^r - b_1^r),$$

and
$$+\frac{1}{r^2}(a_1^r - b_2^r)(a_2^r - b_2^r).$$

It can, in fact, be immediately seen that the condition (2) is merely the one, and the only one, necessitated by the equality of these four expressions. Calling, in fact, either of them $\dfrac{1}{r^2} k_2$, their equality may be expressed by saying that, if $\phi(z) = 0$ be the quadratic of which b_1^r, b_2^r are the roots, then a_1^r, a_2^r are the roots of $\phi(z) = -k_2$, or, in other words, that for all values of z

$$(z - a_1^r)(z - a_2^r) \equiv (z - b_1^r)(z - b_2^r) + k_2.$$

Hence, equating coefficients, we have

$$a_1^r + a_2^r = b_1^r + b_2^r,$$

the sole condition, and $\quad a_1^r a_2^r = b_1^r b_2^r + k_2,$

which determines the most symmetrical value for k_2 when it is satisfied. Thus we have, under the condition (2),

$$\int_0^\infty \int_0^\infty \{f(a_1x_1)\,f(a_2x_2) - f(b_1x_1)\,f(b_2x_2)\}\,\frac{dx_1\,dx_2}{(x_1x_2)^{r+1}} = \frac{A^2}{r^2}\,(a_1^r a_2^r - b_1^r b_2^r) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where A denotes $\displaystyle\int_0^\infty f'(x)\,\frac{dx}{x^r}$.*

A theorem has then been established for the cases $n=1$ and $n=2$, which may now be stated in general, and proved to hold for all cases by the method of mathematical induction; viz.:

If $f(x)$ be such a function and r such a constant that $\displaystyle\int_0^\infty f'(x)\,\frac{dx}{x^r}$ is of finite or vanishing value A, and if the constants $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ be connected, and k_n determined in terms of them by the n conditions which express the identity

$$(z-a_1^r)(z-a_2^r)\ldots(z-a_n^r) \equiv (z-b_1^r)(z-b_2^r)\ldots(z-b_n^r) + (-1)^n k_n \ldots(5);$$

in other words, if $a_1^r, a_2^r, \ldots a_n^r$ and $b_1^r, b_2^r, \ldots b_n^r$ be the roots of two equations of the n^{th} degree which differ only in their constant terms, and if k_n be the difference with proper sign of these constant terms, then

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \{f(a_1x_1)\,f(a_2x_2)\ldots f(a_nx_n) - f(b_1x_1)\,f(b_2x_2)\ldots f(b_nx_n)\} \times \frac{dx_1\,dx_2\ldots dx_n}{(x_1x_2\ldots x_n)^{r+1}} = \frac{A^n}{r^n}\,k_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

---

* The reasoning above is inapplicable to the limiting case when $r=0$. Indeed, in that case, the equation (2) is nothing more than an obvious identity, and must, it is clear, be replaced by the one obtained by taking the limit when r vanishes of

$$\frac{a_1^r - b_1^r}{r} + \frac{a_2^r - b_2^r}{r} = 0, \quad i.e., \quad \text{by } \log a_1 + \log a_2 = \log b_1 + \log b_2.$$

Similar independent processes would have to be gone through at all subsequent steps. For this reason results (4), (6), (12), (13), &c., though correct for all non-vanishing values of r, do not stand the test of proceeding to the limit.

The necessary modifications of the work for the limiting case are, however, with ease developed. The results are already known (see Mr. Leudesdorf's paper), and are, that the coefficient of A^n in (6), or of the multiple integral on the right in (12), has to be replaced by

$$\log a_1 \log a_2 \ldots \log a_n - \log b_1 \log b_2 \ldots \log b_n,$$

which call k_n', the connecting conditions being those which express the identity of

$$(z-\log a_1)\,(z-\log a_2)\ldots(z-\log a_n)$$

with $\quad (z-\log b_1)\,(z-\log b_2)\ldots(z-\log b_n) + (-1)^n k_n'.$

The method to be followed is the usual one of assuming this for the value n and deducing it for $n+1$. Having succeeded in this, since it has been shown true for $n=2$, it will have been proved universally.

Take then the $(n+1)$-tuple integral

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \{f(a_1x_1)f(a_2x_2)\dots f(a_{n+1}x_{n+1}) - f(b_1x_1)f(b_2x_2)\dots f(b_{n+1}x_{n+1})\} \times \frac{dx_1dx_2\dots dx_{n+1}}{(x_1x_2\dots x_{n+1})^{r+1}}.$$

Introducing $n-1$ constants, at present arbitrary, $c_2, \dots c_n$, we write it

$$\int_0^\infty f(a_1x_1)\frac{dx_1}{x_1^{r+1}} \int_0^\infty \int_0^\infty \dots \int_0^\infty \{f(a_2x_2)\dots f(a_nx_n)f(a_{n+1}x_{n+1}) - f(c_2x_2)\dots f(c_nx_n)f(b_{n+1}x_{n+1})\} \times \frac{dx_2\dots dx_ndx_{n+1}}{(x_2\dots x_nx_{n+1})^{r+1}}$$

$$+\int_0^\infty f(b_{n+1}x_{n+1})\frac{dx_{n+1}}{x_{n+1}^{r+1}} \int_0^\infty \int_0^\infty \dots \int_0^\infty \{f(a_1x_1)f(c_2x_2)\dots f(c_nx_n) - f(b_1x_1)f(b_2x_2)\dots f(b_nx_n)\} \times \frac{dx_1dx_2\dots dx_n}{(x_1x_2\dots x_n)^{r+1}},$$

which, on the supposition that the theorem holds for the value n,

$$= \frac{A^n}{r^n}\left\{k_n\int_0^\infty f(a_1x)\frac{dx}{x^{r+1}} + k_n'\int_0^\infty f(b_{n+1}x)\frac{dx}{x^{r+1}}\right\},$$

provided the $2n$ conditions be satisfied which express the identities

$$(z-a_2^r)\dots(z-a_n^r)(z-a_{n+1}^r) \equiv (z-c_2^r)\dots(z-c_n^r)(z-b_{n+1}^r) + (-1)^n k_n \quad \dots\dots\dots\dots(7),$$

$$(z-a_1)^r(z-c_2^r)\dots(z-c_n^r) \equiv (z-b_1^r)(z-b_2^r)\dots(z-b_n^r) + (-1)^n k_n' \dots(8),$$

and which becomes, if the one additional relation

$$k_n + k_n' = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(9),$$

holds,

$$\frac{A^n}{r^n}k_n\int_0^\infty \{f(a_1x) - f(b_{n+1}x)\}\frac{dx}{x^{r+1}},$$

i.e., by (1),

$$\frac{A^{n+1}}{r^{n+1}}k_n(a_1^r - b_{n+1}^r) \dots\dots\dots\dots\dots\dots\dots\dots(10).$$

On the whole, the conditions (7, 8, 9) under which this has been arrived at are in number $2n+1$ in the $3n+3$ constants $a_1, \dots a_{n+1}$, $b_1, \dots b_{n+1}$, $c_2, \dots c_n$, k_n, k_n'. The result of removing n of these, k_n' and the c's, as we wish to do, must be to leave $n+1$ conditions in the remaining $2n+3$. Now, these $n+1$ are easily exhibited by multiplying identities (7) and (8) by $z-a_1^r$ and $z-b_{n+1}^r$, respectively, and adding, so as

to get $(z-a_1^r)(z-a_2^r)\dots(z-a_{n+1}^r) \equiv (z-b_1^r)(z-b_2^r)\dots(z-b_{n+1}^r)$
$+(-1)^n\{k_n(z-a_1^r)+k_n'(z-b_{n+1}^r)\}$
$\equiv (z-b_1^r)(z-b_2^r)\dots(z-b_{n+1}^r)+(-1)^{n+1}k_n(a_1^r-b_{n+1}^r)$,

by the use of (9).

The form of result (10) for the $n+1$-tuple integral, and the $n+1$ conditions expressing this identity found as necessary for its accuracy on the supposition of the soundness of the theorem for n-tuple integrals, are then exactly the form and the $n+1$ conditions which suffice as above to express the theorem for the case of $n+1$. n of the conditions are relations proper in the a's and b's, the $(n+1)^{\text{th}}$ determining the $k_n(a_1^r-b_{n+1}^r)$, or, to use our earlier notation, the k_{n+1}, in terms of them, viz., $k_{n+1}=a_1^r a_2^r\dots a_{n+1}^r-b_1^r b_2^r\dots b_{n+1}^r$.

The theorem is then, by the usual reasoning of mathematical induction, completely established.

The following is the transformation of the theorem first proved to which we are led by the substitution $x=e^y$, $a=e^\alpha$, &c. The independent development of it is possibly even less cumbrous than that gone through above. *If the integral* $\int_{-\infty}^{\infty} e^{-ry}\varphi'(y)\,dy$, *where r is a real constant, have a finite value A, and, if the conditions be satisfied which express the identity*

$$(z-e^{r\alpha_1})(z-e^{r\alpha_2})\dots(z-e^{r\alpha_n}) \equiv (z-e^{r\beta_1})(z-e^{r\beta_2})\dots(z-e^{r\beta_n})+(-1)^n k_n',$$

then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{-r(y_1+y_2+\dots+y_n)}\left\{\prod_{s=1}^{s=n}\varphi(y_s+\alpha_s)-\prod_{s=1}^{s=n}\varphi(y_s+\beta_s)\right\} \times dy_1 dy_2\dots dy_n$$
$$=\frac{A^n}{r^n}k_n'=\frac{A^n}{r^n}\left\{e^{r(\alpha_1+\alpha_2+\dots+\alpha^n)}-e^{r(\beta_1+\beta_2+\dots+\beta_n)}\right\}\dots(11).$$

From the sub-class we now proceed to

II. *The General Integral*

$$\int_0^\infty\int_0^\infty\dots\int_0^\infty\{S(a_1x_1,\ a_2x_2,\ \dots\ a_nx_n)-S(b_1x_1,\ b_2x_2,\ \dots\ b_nx_n)\}\frac{dx_1dx_2\dots dx_n}{(x_1x_2\dots x_n)^{r+1}},$$

where S is any suitable symmetric function.

The double integral

$$\int_0^\infty\int_0^\infty\{S(a_1x_1,\ a_2x_2)-S(b_1x_1,\ b_2x_2)\}\frac{dx_1dx_2}{(x_1x_2)^{r+1}}$$
$$\equiv\int_0^\infty\int_0^\infty\{S(a_1x_1,\ a_2x_2)-S(a_1x_1,\ b_2x_2)+S(a_1x_1,\ b_2x_2)-S(b_1x_1,\ b_2x_2)\} \times\frac{dx_1dx_2}{(x_1x_2)^{r+1}}\};$$

and this, by two uses of (1), the applicability of which would seem

to require that $S(x, c)$ be, for all values of c, a function of x such as the $f(x)$ of that equality,

$$= \frac{a_2^r - b_2^r}{r} \int_0^\infty \int_0^\infty \frac{d}{dx_2} S(a_1 x_1, x_2) \frac{dx_1 dx_2}{x_1^{r+1} x_2^r} + \frac{a_1^r - b_1^r}{r} \int_0^\infty \int_0^\infty \frac{d}{dx_1} S(x_1, b_2 x_2) \frac{dx_1 dx_2}{x_1^r x_2^{r+1}}$$

$$= -\frac{a_1^r - b_1^r}{r} \int_0^\infty \int_0^\infty \frac{d}{dx_2} \{S(a_1 x_1, x_2) - S(b_2 x_1, x_2)\} \frac{dx_1 dx_2}{x_1^{r+1} x_2^r},$$

provided the condition $a_1^r + a_2^r = b_1^r + b_2^r$ be satisfied,

$$= -\frac{(a_1^r - b_2^r)(a_1^r - b_2^r)}{r^2} \int_0^\infty \int_0^\infty \frac{d^2}{dx_1 dx_2} S(x_1, x_2) \frac{dx_1 dx_2}{(x_1 x_2)^r},$$

by a further application of (1) which, since the operation $\frac{d}{dx_2}$ may be considered subsequent to the integration with respect to x_1, appears to necessitate no further limitation of the forms of function S. The work of modifying the factor multiplying the integral in this result proceeds exactly as the corresponding reduction which produced (4), and need not be repeated.

The general theorem to which we are led may now be stated: viz., *If* $S(x_1, x_2, \ldots x_n)$ *be a symmetric function of its* n *arguments, restricted in form by a condition, which would seem to be that for all values of* $n-1$ *of these arguments, it is such a function* S *of the* n^{th} *argument* x *that for a certain value of* r $\int_0^\infty \frac{dS}{dx} \cdot \frac{dx}{x^r}$ *is finite, then,* r *having that value,*

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \{S(a_1 x_1, a_2 x_2, \ldots a_n x_n) - S(b_1 x_1, b_2 x_2, \ldots b_n x_n)\} \frac{dx_1 dx_2 \ldots dx_n}{(x_1 x_2 \ldots x_n)^{r+1}}$$

$$= \frac{k_n}{r^n} \int_0^\infty \int_0^\infty \ldots \int_0^\infty \frac{d^n}{dx_1 dx_2 \ldots dx_n} S(x_1, x_2, \ldots x_n) \frac{dx_1 dx_2 \ldots dx_n}{(x_1 x_2 \ldots x_n)^r} \ldots (12),$$

if $a_1, a_2, \ldots a_n$, $b_1, b_2, \ldots b_n$ *be constants connected, and* k_n *one determined in terms of them, by the* n *conditions which express the identity*

$$(z - a_1^r)(z - a_2^r) \ldots (z - a_n^r) \equiv (z - b_1^r)(z - b_2^r) \ldots (z - b_n^r) + (-1)^n k_n \ldots (5).$$

The derivation of this from the case of $n = 2$ by mathematical induction proceeds exactly as in the case of the sub-theorem already demonstrated, as far as conditions are concerned. It will be well, however, to exhibit in outline the deduction of the integral form for case $n+1$ from that for case n. The method is, as before, to introduce $n-1$ new constants $c_2, c_3, \ldots c_n$. We thus have

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty \{S(a_1 x_1, \ldots a_n x_n, a_{n+1} x_{n+1}) - S(b_1 x_1, \ldots b_n x_n, b_{n+1} x_{n+1})\}$$

$$\times \frac{dx_1 dx_2 \ldots dx_n dx_{n+1}}{(x_1 \ldots x_n x_{n+1})^{r+1}}$$

$$= \int_0^\infty \int_0^\infty \dots \int_0^\infty \{S(a_1x_1, a_2x_2, \dots a_nx_n, a_{n+1}x_{n+1}) - S(a_1x_1, c_2x_2, \dots c_nx_n, b_{n+1}x_{n+1})\} \times \frac{dx_1dx_2 \dots dx_{n+1}}{(x_1x_2 \dots x_{n+1})^{r+1}}$$

$$+ \int_0^\infty \int_0^\infty \dots \int_0^\infty \{S(a_1x_1, c_2x_2, \dots c_nx_n, b_{n+1}x_{n+1}) - S(b_1x_1, b_2x_2, \dots b_nx_n, b_{n+1}x_{n+1})\} \times \frac{dx_1dx_2 \dots dx_{n+1}}{(x_1x_2 \dots x_{n+1})^{r+1}}$$

$$= \frac{k_n}{r^n} \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_2 \dots dx_{n+1}} S(a_1x_1, x_2, \dots x_{n+1}) \frac{dx_1dx_2 \dots dx_{n+1}}{x_1^{r+1}(x_2 \dots x_{n+1})^r}$$

$$+ \frac{k'_n}{r^n} \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1 \dots dx_n} S(x_1, \dots x_n, b_{n+1}x_{n+1}) \frac{dx_1dx_2 \dots dx_{n+1}}{(x_1 \dots x_n)^r x_{n+1}^{r+1}},$$

by two applications of the theorem for the case n under the proper conditions; and, further, by cyclical interchange of the suffixes of the variables, and an application of (1) under the last condition $k_n+k'_n=0$,

$$= k_n(a_1^r - b_{n+1}^r) \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1dx_2 \dots dx_{n+1}} S(x_1, x_2, \dots x_{n+1}) \frac{dx_1dx_2 \dots dx_{n+1}}{(x_1x_2 \dots x_{n+1})^r},$$

the limitation of $S(x_1, x_2, \dots x_{n+1})$ implied being, it would seem, that stated above, since the operation $\frac{d^n}{dx_2 \dots dx_{n+1}}$ may be considered subsequent to the integration with regard to x_1. The remainder of the reasoning is exactly that by which (6) was established.

The companion theorem, derived from the form of (1) which gave us (11), may be stated :—*If* $S(y_1, y_2, \dots y_n)$ *be such a symmetric function of its n arguments as to be, whatever be any* $n-1$ *of them, a function of the* n^{th} *of the nature of the* $\phi(y)$ *introduced in* (11), *then, under conditions which express the identity*

$$(z-e^{r\alpha_1})(z-e^{r\alpha_2}) \dots (z-e^{r\alpha_n}) \equiv (z-e^{r\beta_1})(z-e^{r\beta_2}) \dots (z-e^{r\beta_n}) + (-1)^n k'_n,$$

the equality holds—

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \dots \int_{-\infty}^\infty e^{-r(y_1+y_2+\dots+y_n)} \{S(y_1+\alpha_1, y_2+\alpha_2, \dots y_n+\alpha_n) - S(y_1+\beta_1, y_2+\beta_2, \dots y_n+\beta_n)\} \times dy_1dy_2 \dots dy_n$$

$$= \frac{k'_n}{r^n} \int_{-\infty}^\infty \int_{-\infty}^\infty \dots \int_{-\infty}^\infty e^{-r(y_1+y_2+\dots+y_n)} \frac{d^n}{dy_1dy_2 \dots dy_n} S(y_1, y_2, \dots y_n) \times dy_1dy_2 \dots dy_n \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(13).$$

The apparent stringency of the limitations to the forms of symmetric function in these two companion forms of result is great, and

may probably be modified. It will indeed be strange if in each case what is necessary for the result arrived at to be true is not in reality exactly what is required for the integral on the right-hand side to be intelligible and finite. It may be remarked that the general results give correctly the first obtained ones (5) and (11), and also, with the proper modification of the factor k_n or k'_n according to the note on page 15, the special ones for the case $r = 0$ treated of in my own and Mr. Leudesdorf's former papers.

A new point of view of the whole subject of Frullanian integrals, which may possibly indicate its real extensiveness, is suggested by the forms of the general results (12) and (13). Taking the former, and applying to it the successive differentiation $\dfrac{d^n}{da_1 da_2 \dots da_n}$, we arrive at a result which expresses that the integral on the right is one to which it is lawful to apply the transformation which replaces $x_1, x_2, \dots x_n$ by $a_1 x_1, a_2 x_2, a_n x_n$. From this starting-point the results obtained might possibly have been arrived at with greater elegance than has here been done. The all-including result would be obtained by starting from the most general equality of transformation which could be proved of the form

$$\int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{da_1 da_2 \dots da_n} F(a_1 x_1, a_2 x_2, \dots a_n x_n) \frac{dx_1 dx_2 \dots dx_n}{\phi(x_1, x_2, \dots x_n)}$$
$$= \psi(a_1, a_2, \dots a_n) \int_0^\infty \int_0^\infty \dots \int_0^\infty \frac{d^n}{dx_1 dx_2 \dots dx_n} F(x_1, x_2, \dots x_n) \frac{dx_1 dx_2 \dots dx_n}{\chi(x_1, x_2, \dots x_n)},$$

and proceeding by integration with regard to the constants.

On Symmetric Functions, and in particular on certain Inverse Operators in connection therewith. By Captain P. A. MacMahon, R.A.

[*Read Nov. 8th*, 1883.]

1. The present paper is more especially concerned with the non-unitary symmetric functions (that is, with those whose partitions contain no unit), their calculation and the development of their properties; for the reason that it has been recently shown (*vide* the author's paper in Vol. vi., No. 2, *American Journal of Mathematics*) that they are, in fact, seminvariants of an allied quantic, and the whole series contains potentially the complete solution of the syzygies which exist between the sources of covariants of binary quantics.

The equation considered is in every case

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - a_3 x^{n-3} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and, as much use will be made of Mr. Hammond's "Differential Operators," reference is made to his paper in the *Proceedings of the London Mathematical Society* for February, 1883, Vol. xiv., pp. 119—129; the operator most employed is

$$d_\lambda = \frac{d}{da_\lambda} + a_1 \frac{d}{da_{\lambda+1}} + a_2 \frac{d}{da_{\lambda+2}} + \ldots,$$

d_λ being defined by

$$d_\lambda = [\lambda],$$

where the partitions in [ ] refer to the symmetric functions of the roots of the equation

$$1 - y^{-1} D_1 + y^{-2} D_2 - y^{-3} D_3 + \ldots = 0;$$

and the law of operation of D on the partitions of the symmetric functions of the roots of (1) is, from Mr. Hammond's paper, as follows:—

$$D_\kappa (\lambda^l . \mu^m . \nu^n \ldots) = 0, \quad D_\lambda (\lambda^l . \mu^m . \nu^n \ldots) = (\lambda^{l-1} . \mu^m . \nu^n \ldots), \quad D_\lambda (\lambda) = 1.$$

If the symmetric functions be rendered homogeneous by multiplying each term by a_0 raised to the necessary power, then

$$d_\lambda = a_0 \frac{d}{da_\lambda} + a_1 \frac{d}{da_{\lambda+1}} + a_2 \frac{d}{da_{\lambda+2}} + \ldots \quad \ldots\ldots\ldots (2).$$

Putting herein $a_0 = 0$, the character of the operator is not altered, and, in fact, the change is merely equivalent to a unit increase of suffixes throughout; this shows that the terms of highest degree in all symmetric functions of (1) are simply symmetric functions of the equation

$$x^n - a_2 x^{n-1} + a_3 x^{n-3} - a_4 x^{n-4} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

a_1 being now the factor introduced for sake of homogeneity. Thus, every known symmetric function furnishes the terms of highest degree in an infinite number of higher symmetric functions by the simple process of increasing each suffix by unity, and then multiplying the whole by the necessary power of a_1; for example, if partitions in (( )) refer to equation (3), then the terms of highest degree in $(\lambda^l . \mu^m . \nu^n \ldots)$ are

$$a_1^{\lambda-\mu} ((\mu^m . \nu^n \ldots)),$$

that is, from symmetric function $(\mu^m . \nu^n \ldots)$ we can, by giving λ different values $>$ or $= \mu$, derive the highest terms in an infinite number of others; *e.g.*, since

$$(3^2 2^2) = a_2 a_4^2 - 2a_2 a_3 a_5 - a_1 a_4 a_5 + 5a_5 + 2a_2^2 a_6 + 3a_1 a_3 a_6 - 9a_4 a_6 - 7a_1 a_2 a_7 + 6a_3 a_7 + 7a_1^2 a_8 + a_2 a_8 - 15a_1 a_9 + 15a_{10},$$

we find

$$(3^3 2^2) = a_3a_5^2 - 2a_3a_4a_6 - a_2a_5a_6 + 5a_1a_6^2 + 2a_3^2a_7 + 3a_2a_4a_7$$
$$-9a_1a_5a_7 - 7a_2a_3a_8 + 6a_1a_4a_8 + 7a_2^2a_9 + a_1a_3a_9 - 15a_1a_2a_{10}$$
$$+15a_1^2a_{11} + \text{terms of lower degree},$$

and so on in every case, the terms of highest degree in all symmetric functions of a given weight may be simply derived from those of lower weight.

It follows hence that the performance of the differential operation denoted by d_λ may be somewhat simplified; for, when $a_0 = 0$, let d_λ become d'_λ, so that

$$d_\lambda = \frac{d}{da_\lambda} + d'_\lambda,$$

and let the subject operated upon be

$$a_1^\kappa A + B,$$

wherein $a_1^\kappa A$ comprises the terms of highest degree then if, as is generally the case, the partition of the subject contain no partition of λ,

$$\left(\frac{d}{da_\lambda} + d'_\lambda\right)(a_1^\kappa A + B) = 0;$$

then $\lambda \neq 1, \quad a_1^\kappa \frac{dA}{d\lambda} + d_\lambda B = 0$

and $\lambda = 1, \quad a^{\kappa-1}\left(a_1 \frac{dA}{da_1} + \kappa A\right) + d_1 B = 0$ } .....(4),

A being of course already known.

Suppose, for instance, the subject be $(\mu \,.\, \nu \,.\, \pi \ldots)$; if the partition contain no partition of λ,

$$a_1^{\mu-\nu}\frac{d}{da_\lambda}((\nu \,.\, \pi \ldots)) + d_\lambda B = 0,$$

or

$$a_1^{\mu-\nu}\left(\left(\frac{d}{da_{\lambda-1}}(\nu \,.\, \pi \ldots)\right)\right) + d_\lambda B = 0,$$

2. A non-unitary symmetric function is completely given by its non-unitary portion; this follows at once from the fact that the complete expression of a seminvariant can be derived from its non-unitary portion, by substituting for the coefficients therein, the corresponding coefficients of Professor Cayley's generalized canonical form (*vide* Vol. xix., No. 76, *Quarterly Journal of Mathematics*); the corresponding theorem in symmetric functions is as follows, viz.:—forming the

equation $x^n + A_2 x^{n-2} - A_3 x^{n-3} + A_4 x^{n-4} - \ldots = 0$ .....(5),

in which A_r is formed from the non-unitary symmetric functions of weight r, and is such that on putting therein $a_1 = 0$ it reduces to a_r; then we find

$$A_2 = a_2 - \frac{1}{2} a_1^2,$$

$$A_3 = a_3 - a_1 a_2 + \frac{2}{3!} a_1^3,$$

$$A_4 = a_4 - a_1 a_3 + \frac{1}{2!} a_1^2 a_2 - \frac{3}{4!} a_1^4,$$

$$A_5 = a_5 - a_1 a_4 + \frac{1}{2!} a_1^2 a_3 - \frac{1}{3!} a_1^3 a_2 + \frac{4}{5!} a_1^5,$$

$$A_6 = a_6 - a_1 a_5 + \frac{1}{2!} a_1^2 a_4 - \frac{1}{3!} a_1^3 a_3 + \frac{1}{4!} a_1^4 a_2 - \frac{5}{6!} a_1^6,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$A_r = a_r - a_1 a_{r-1} + \frac{1}{2!} a_1^2 a_{r-2} - \frac{1}{3!} a_1^3 a_{r-3} + \frac{1}{4!} a_1^4 a_{r-4} - \ldots$$
$$+ (-)^r \frac{1}{(r-2)!} a_1^{r-2} a_2 + (-)^{r+1} \frac{r-1}{r!} a_1^r.$$

Now the non-unitary symmetric functions of (5) are also non-unitary functions of (1), and the equations (1) and (5) become the same on putting therein $a_1 = 0$; consequently the non-unitary symmetric functions of (1) and (5) must be identical, and if

$$\phi(a_2, a_3, a_4 \ldots)$$

be the non-unitary portion of a function of (1),

$$\phi(A_2, A_3, A_4 \ldots)$$

must be its complete expression; now $\lambda > 1$,

$$d_\lambda (a_1 = 0) = \frac{d}{da_\lambda} + a_2 \frac{d}{da_{\lambda+2}} + a_3 \frac{d}{da_{\lambda+3}} + \ldots,$$

and the operators included in this formula may be employed to calculate separately the non-unitary portions of the symmetric functions of (1).

*3. In this and the succeeding section, Mr. Hammond's operators (see *Proceedings*, Vol. xiv., p. 119), included in d_λ, are regarded from an inverse point of view; *i.e.*, instead of regarding d_λ as a function of certain differential coefficients $\frac{d}{da_\lambda}, \frac{d}{da_{\lambda+1}}, \ldots$, we look upon $\frac{d}{da^\lambda}$ as a

* The proofs of §§ 3 and 4 have in great measure been communicated to me by Mr. Hammond.

function of the operators d_λ, $d_{\lambda+1}$, $d_{\lambda+2}$, ..., which leads us at once to a system of inverse operators, enabling us to proceed by an operation from a lower to a higher symmetric function.

First, if α, β, γ, ... be the roots of

$$x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots = 0,$$

$$\frac{1}{(1-\alpha x)(1-\beta x)(1-\gamma x)\ldots} = 1 + H_{(1)}x + H_{(2)}x^2 + H_{(3)}x^3 + \ldots = u \text{ suppose},$$

$$= \frac{1}{1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots},$$

$H_{(r)}$ being the total symmetric function of weight r. Therefore

$$(1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots)(1 + H_{(1)}x + H_{(2)}x^2 + H_{(3)}x^3 + \ldots) = 1;$$

equating coefficients,

$$H_{(1)} - a_1 = 0,$$
$$H_{(2)} - a_1 H_{(1)} + a_2 = 0,$$
$$H_{(3)} - a_1 H_{(2)} + a_2 H_{(1)} - a_3 = 0,$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

generally $$H_{(r)} - a_1 H_{(r-1)} + a_2 H_{(r-2)} - \ldots + (-)^r a_r = 0 \ldots\ldots\ldots (6).$$

Now, these equations may be solved either for a_r or H_r, giving either

$$a_r = \begin{vmatrix} H_{(1)}, & H_{(2)}, & H_{(3)}, & \ldots, & H_{(r-1)}, & H_{(r)} \\ 1, & H_{(1)}, & H_{(2)}, & \ldots, & H_{(r-2)}, & H_{(r-1)} \\ & 1, & H_{(1)}, & \ldots, & H_{(r-3)}, & H_{(r-2)} \\ & & 1, & \ldots, & H_{(r-4)}, & H_{(r-3)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & & 1, & H_{(1)} \end{vmatrix},$$

or the corresponding formula with a and H interchanged. Consequently, if $$H_{(r)} = \phi(a_1, a_2, a_3, \ldots, a_r),$$

then $$a_r = \phi(H_{(1)}, H_{(2)}, H_{(3)}, \ldots, H_{(r)}).$$

Now, considering the system of equations

$$\frac{d}{da_1} + a_1 \frac{d}{da_2} + a_2 \frac{d}{da_3} + \ldots = d_1,$$
$$\frac{d}{da_2} + a_1 \frac{d}{da_3} + \ldots = d_2,$$
$$\frac{d}{da_3} + \ldots = d_3,$$
$$\ldots = \ldots,$$

and solving for $\frac{d}{da_\lambda}$, we obtain

$$\begin{vmatrix} 1, & a_1, & a_2, & \ldots \\ & 1, & a_1, & \ldots \\ & & 1, & \ldots \\ \ldots & \ldots & \ldots & \end{vmatrix} \frac{d}{da_\lambda} = \begin{vmatrix} d_\lambda, & a_1, & a_2, & a_3, & \ldots \\ d_{\lambda+1}, & 1, & a_1, & a_2, & \ldots \\ d_{\lambda+2}, & . & 1, & a_1, & \ldots \\ d_{\lambda+3}, & . & . & 1, & \ldots \\ & \ldots & \ldots & \ldots & \ldots \end{vmatrix};$$

whence $$\frac{d}{da_\lambda} = d_\lambda - H_1 d_{\lambda+1} + \begin{vmatrix} a_1, & a_2 \\ 1, & a_1 \end{vmatrix} d_{\lambda+2} - \begin{vmatrix} a_1, & a_2, & a_3 \\ 1, & a_1, & a_2 \\ & 1, & a_1 \end{vmatrix} d_{\lambda+3} + \ldots,$$

that is, $$\frac{d}{da_\lambda} = d_\lambda - H_{(1)} d_{\lambda+1} + H_{(2)} d_{\lambda+2} - H_{(3)} d_{\lambda+3} + \ldots \quad \ldots\ldots\ldots(7),$$

wherein if the subject be $(\lambda^l . \mu^m . \nu^n, \ldots)$,

$$d_0 (\lambda^l . \mu^m . \nu^n \ldots) = \lambda (\lambda^l . \mu^m . \nu^n \ldots),$$

which is merely Euler's theorem of homogeneous functions.

It is necessary at this point to prove certain properties of the total symmetric function $H_{(r)}$.

Since $$u = \frac{1}{(1-\alpha x)(1-\beta x)\ldots} = 1 + H_{(1)} x + H_{(2)} x^2 + \ldots,$$

$$\frac{du}{dx} = H_{(1)} + 2H_{(2)} x + 3H_{(3)} x^2 + \ldots,$$

and, differentiating u logarithmically,

$$\frac{1}{u}\frac{du}{dx} = \frac{\alpha}{1-\alpha x} + \frac{\beta}{1-\beta x} + \ldots = S_1 + S_2 x + S_3 x^2 + \ldots;$$

whence

$$(1 + H_{(1)} x + H_{(2)} x^2 + \ldots)(S_1 + S_2 x + S_3 x^2 + \ldots) = H_{(1)} + 2H_{(2)} x + 3H_{(3)} x^2 + \ldots,$$

and, equating coefficients, there follows the system

$$\left.\begin{aligned} H_{(1)} &= S_1 \\ 2H_{(2)} &= S_2 + H_{(1)} S_1, \\ 3H_{(3)} &= S_3 + H_{(1)} S_2 + H_{(2)} S_1 \\ \ldots\ \ldots\ \ldots

Solving these equations for S_r and for $H_{(r)}$, we get

$$S_r = (-)^{r+1} \begin{vmatrix} H_{(1)}, & H_{(2)}, & H_{(3)}, & \ldots, & H_{(r-1)}, & r\,.\,H_{(r)} \\ 1, & H_{(1)}, & H_{(2)}, & \ldots, & H_{(r-2)}, & r-1\,.\,H_{(r-1)} \\ 0, & 1, & H_{(1)}, & \ldots, & H_{(r-3)}, & r-2\,.\,H_{(r-2)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ . & . & . & \ldots, & 1, & H_{(1)} \end{vmatrix} \quad \ldots\ldots(9),$$

$$r!\,H_{(r)} = \begin{vmatrix} S_1, & S_2, & S_3, & \ldots, & S_r \\ -1, & S_1, & S_2, & \ldots, & S_{r-1} \\ . & -2, & S_1, & \ldots, & S_{r-2} \\ . & . & -3, & \ldots, & S_{r-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ & & & -(r-1), & S_1 \end{vmatrix} \quad \ldots\ldots(10);$$

consequently, any expression of a symmetric function is unaltered by the substitution of H for a, if, when expressed in terms of the sums of powers of the roots, each term contains only sums of odd powers and an even number of sums of even powers. The symmetric functions, when expressed in terms of $H_{(1)}, H_{(2)}, H_{(3)} \ldots$, appear to obey the law of symmetry; that is, the coefficient of $H_{(\lambda)}^{l_1} . H_{(\mu)}^{m_1} \ldots$ in symmetric function $(\lambda_1^{l_1} . \mu_1^{m_1} \ldots)$ is equal to the coefficient of $H_{(\lambda_1)}^{l} . H_{(\mu_1)}^{m_1} \ldots$ in symmetric function $(\lambda^l . \mu^m \ldots)$; thus the table for weight 4 is

	$H_{(1)}^4$	$H_{(1)}^2 H_{(2)}$	$H_{(2)}^2$	$H_{(1)} H_{(3)}$	$H_{(4)}$
(4)	−1	+4	−2	−4	+4
(31)	+2	−7	+2	+7	−4
(2^2)	+1	−4	+3	+2	−2
(21^2)	−3	+10	−4	−7	+4
(1^4)	+1	−3	+1	+2	−1

...(11).

Now, referring to the set of equations (7), if S_m be the subject of operation, we find

$$\frac{d}{da_{m-r}} S_m = (-)^{m+r+1} m H_{(r)} \quad \ldots\ldots(12),$$

and

$$H_{(m-1)} = \frac{1}{m} \frac{d}{da_1} S_m \quad \ldots\ldots(13).$$

Now, since the general term in S_m is

$$\frac{(-)^{k+n} (k-1)!\, n}{a!\, b!\, c! \ldots l!} a_1^a a_2^b a_3^c \ldots a_n^l \quad \ldots\ldots(14),$$

in which k is the degree of the term, we deduce without difficulty that the general term in $H_{(n)}$ is

$$\frac{(-)^{k+n}\,k\,!}{a\,!\,b\,!\,c\,!\,\ldots\,l\,!}\,a_1^a a_2^b a_3^c \ldots a_n^l \quad \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

This gives at once a curious theorem, viz. :—

"If in the expression of S_n in terms of the coefficients we multiply each term by the degree of the term, and then divide the whole expression by the degree of the expression, we obtain the sum of all the homogeneous symmetric functions of weight n; and conversely, if in the expression of $H_{(n)}$ in terms of the coefficients we multiply throughout by the degree of the expression, and then divide each term by the degree of the term, we obtain the expression of S_n."

Thus

$$\begin{aligned}
H_{(n)} = {} & a^n - (n-1)\,a_1^{n-2}a_2 + (n-2)\,a_1^{n-3}a_3 - (n-3)\,a_1^{n-4}\left\{a_4 - (n-2)\frac{a_2^2}{2!}\right\} \\
& + (n-4)\,a_1^{n-5}\{a_5 - (n-3)\,a_2a_3\} - (n-5)\,a_1^{n-6}\left\{a_6 - (n-4)\left(a_2a_4 + \frac{a_3^2}{2!}\right)\right. \\
& \left. + (n-3)(n-4)\frac{a_2^3}{3!}\right\} + (n-6)\,a_1^{n-7}\left\{a_7 - (n-5)(a_2a_5 + a_3a_4)\right. \\
& \left. + (n-4)(n-5)\frac{a_2^2a_3}{2!}\right\} - (n-7)\,a_1^{n-8}\left\{a_8 - (n-6)\left(a_2a_6 + a_3a_5 + \frac{a_4^2}{2!}\right)\right. \\
& \left. + (n-5)(n-6)\left(\frac{a_2^2a_4}{2!} + \frac{a_2a_3^2}{2!}\right) - (n-4)(n-5)(n-6)\frac{a_2^4}{4!}\right\} \\
& + (n-8)\,a_1^{n-9}\left\{a_9 - (n-7)(a_2a_7 + a_3a_6 + a_4a_5)\right. \\
& \left. + (n-6)(n-7)\left(\frac{a_2^2a_5}{2!} + a_2a_3a_4 + \frac{a_3^3}{3!}\right) - (n-5)(n-6)(n-7)\frac{a_2^3a_3}{3!}\right\} \\
& - (n-9)\,a_1^{n-10}\left\{a_{10} - (n-8)\left(a_2a_8 + a_3a_7 + a_4a_6 + \frac{a_5^2}{2!}\right)\right. \\
& \left. + (n-7)(n-8)\left(\frac{a_2^2a_6}{2!} + a_2a_3a_5 + \frac{a_2a_4^2}{2!} + \frac{a_3^2a_4}{2!}\right)\right. \\
& \left. - (n-6)(n-7)(n-8)\left(\frac{a_2^3a_4}{3!} + \frac{a_2^2a_3^2}{2!\,2!}\right)\right. \\
& \left. - (n-5)(n-6)(n-7)(n-8)\frac{a_2^5}{5!}\right\} + \&c. \quad \ldots\ldots\ldots\ldots\ldots\ldots(16).
\end{aligned}$$

The process, indicated above, by which S_n is transformed into $H_{(n)}$, will, in what follows, be denoted by $\left(\frac{\delta}{\Delta}\right)$, and the reverse process

by $\left(\frac{\Delta}{\delta}\right)$; thus $\left(\frac{\delta}{\Delta}\right) S_n = H_{(n)}$ .....(17),

$$\left(\frac{\Delta}{\delta}\right) H_{(n)} = S_n \text{ .....(18).}$$

It follows that a large portion of S_{n+1} can be obtained by the use of S_n alone; for from (17) we get $H_{(n)}$, and then from (13)

$$S_{n+1} = (n+1)\int_0^{a_1} H_{(n)}\, da_1 + \text{...} \text{ .....(19),}$$

e.g., $S_3 = a_1^3 - 3a_1a_2 + 3a_3$,

therefore $H_{(3)} = a_1^3 - 2a_1a_2 + a_3$,

therefore $S_4 = 4\left(\frac{1}{4}a_1^4 - a_1^2a_2 + a_1a_3\right) + \text{...}$

The operation denoted by $\left(\frac{\delta}{\Delta}\right)$ is equivalent to another, because, from the last equation of (7), if S_n be the subject, and we introduce a_0 for homogeneity, we find

$$\frac{d}{da_0} S_n = nS_n - nH_{(n)},$$

or $$H_{(n)} = S_n - \frac{1}{n}\frac{d}{da_0} S_n \text{ .....(20),}$$

or $$\left(\frac{\delta}{\Delta}\right) S_n = S_n - \frac{1}{n}\frac{d}{da_0} S_n \text{ .....(21);}$$

the operator $\left(\frac{\delta}{\Delta}\right)$ will be further considered presently.

Reverting now to the general theory, it is clear that the equations (7) enable us to calculate, separately, that portion of a symmetric function which contains any particular coefficient, and in every case by a simple process of integration, and the advantage of this method consists in the fact that the coefficient of a term containing p different coefficients will be subject to $p-1$ independent verifications; in terms whose degree is less than that of the function, a_0 of course takes its place as a coefficient.

The method is illustrated by the calculation of (5^23).

Putting $(5^23) = \phi$,

$\frac{d\phi}{da_{13}} = d_{13}\phi = +13$, therefore $\phi = 13a_{13} + \text{...}$,

$\frac{d\phi}{da_{12}} = -H_{(1)}d_{13}\phi = -13a_1$, therefore $\phi = -13a_1a_{12} + \text{...}$,

$\frac{d\phi}{da_{11}} = +H_{(2)}d_{13}\phi = 13a_1^2 - 13a_2$, therefore $\phi = 13a_1^2a_{11} - 13a_2a_{11} + \text{...}$,

$\frac{d\phi}{da_{10}} = d_{10}\phi - H_{(3)}d_{13}\phi = +5(3) - 13H_{(3)} = -8a_1^3 + 11a_1a_2 + 2a_3$,

therefore $\phi = -8a_1^3a_{10}+11a_1a_2a_{10}-2a_3a_{10}+\ldots,$

and so on ;

$$\frac{d\phi}{da_5} = d_5\phi - H_{(3)}d_8\phi - H_{(5)}d_{10}\phi + H_{(8)}d_{13}\phi$$
$$= 5\,(53) - 8H_{(3)}(5) - 5H_{(5)}(3) + 13H_{(8)};$$

in reducing this, we reject at once all terms which either are, or give rise to, terms above the fourth degree, for ϕ, we know, is only of the fifth degree.

We find
$$\frac{d\phi}{da_5} = 3a_2^4 - 9a_1a_2^2a_3 + 8a_1^2a_3^2 - 4a_2a_3^2 + a_1^2a_2a_4$$
$$+ a_2^2a_4 - 3a_1a_3a_4 - 7a_4^2 - 8a_1^3a_5 + 22a_1a_2a_5$$
$$+ 6a_3a_5 + a_1^2a_6 - 14a_2a_6 - 14a_1a_7 + 27a_8,$$

and therefore

$$\phi = 3a_2^4a_5 - 9a_1a_2^2a_3a_5 + 8a_1^2a_3^2a_5 - 4a_2a_3^2a_5 + a_1^2a_2a_4a_5 + a_2^2a_4a_5 - 3a_1a_3a_4a_5$$
$$- 7a_4^2a_5 - 4a_1^3a_5^2 + 11a_1a_2a_5^2 + 3a_3a_5^2 + a_1^2a_5a_6 - 14a_2a_5a_6 - 14a_1a_5a_7 + 27a_5a_8$$
$$+ \ldots,$$

and so forth.

Finally, we obtain the amply verified result,

$$\begin{aligned}(5^2 3) = {} & a_2^2a_3^3 - 2a_1a_3^4 - 3a_2^3a_3a_4 + 6a_1a_2a_3^2a_4 + 2a_3^3a_4 + 3a_1a_2^2a_4^2 \\ & - 7a_1^2a_3a_4^2 - 4a_2a_3a_4^2 + 7a_1a_4^3 + 3a_2^4a_5 - 9a_1a_2^2a_3a_5 + 8a_1^2a_3^2a_5 \\ & - 4a_2a_3^2a_5 + a_1^2a_2a_4a_5 + a_2^2a_4a_5 - 3a_1a_3a_4a_5 - 7a_4^2a_5 - 4a_1^3a_5^2 \\ & + 11a_1a_2a_5^2 + 3a_3a_5^2 - 3a_1a_2^3a_6 + a_1^2a_2a_3a_6 + 16a_2^2a_3a_6 - 9a_1a_3^2a_6 \\ & + 7a_1^3a_4a_6 - 23a_1a_2a_4a_6 + 11a_3a_4a_6 + a_1^2a_5a_6 - 14a_2a_5a_6 + 13a_1a_6^2 \\ & + 8a_1^2a_2^2a_7 - 13a_2a_7 - 8a_1^3a_3a_7 + 7a_1a_2a_3a_7 - 2a_3^2a_7 + a_1^2a_4a_7 \\ & + 26a_2a_4a_7 - 14a_1a_5a_7 - 13a_6a_7 - 8a_1^3a_2a_8 + 16a_1a_2^2a_8 + 16a_1^2a_3a_8 \\ & - 29a_2a_3a_8 - 14a_1a_4a_8 + 27a_5a_8 + 8a_1^4a_9 - 24a_1^2a_2a_9 + 13a_2^2a_9 \\ & + 11a_1a_3a_9 - 13a_4a_9 - 8a_1^3a_{10} + 11a_1a_2a_{10} + 2a_3a_{10} + 13a_1^2a_{11} \\ & - 13a_2a_{11} - 13a_1a_{12} + 13a_{13} \ldots\ldots\ldots\

and, r being $< m$, $d_r S_{m+1} = 0$, so that

$$PS_{m+1} + \frac{d}{da_{r+1}} S_{m+1} = 0.$$

Thus
$$d_r H_{(m)} = -\frac{1}{m+1} \frac{d}{da_{r+1}} S_{m+1},$$

or, from (12),

$$d_r H_{(m)} = (-)^{r+1} H_{(m-r)} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (23);$$

this formula will be of use in the sequel.

Now, since

$$\frac{1}{u} = 1 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots,$$

we have

$$\frac{1}{u}\frac{du}{dx} = (1 - a_1 x + a_2 x^2 - \ldots)(H_{(1)} + 2H_{(2)} x + 3H_{(3)} x^2 + \ldots)$$
$$= S_1 + S_2 x + S_3 x^2 + \ldots$$

Therefore, equating coefficients of like powers of x,

$$rH_{(r)} - (r-1)\, a_1 H_{(r-1)} + (r-2)\, a_2 H_{(r-2)} - \ldots = S_r,$$

or, changing r into $r+1$, and remembering that

$$H_{(r)} - a_1 H_{(r-1)} + \ldots = 0,$$

we find

$$a_1 H_{(r)} - 2a_2 H_{(r-1)} + 3a_3 H_{(r-2)} - \ldots = S_{r+1} \quad \ldots\ldots\ldots\ldots (24).$$

Again, since $r > 1$,

$$(r \,.\, 1^s) = a_{s+1} S_{r-1} - a_{s+2} S_{r-2} + \ldots + (-)^{r+1} (r+s)\, a_{r+s}$$

(see *Proceedings*, Vol. xiv., p. 122); multiplying this by $(-)^s$ and adding Newton's series

$$S_{r+s} - a_1 S_{r+s-1} + a_2 S_{r+s-2} - \ldots + (-)^{r+s} (r+s)\, a_{r+s},$$

we obtain

$$(-)^s (r \,.\, 1^s) = S_{r+s} - a_1 S_{r+s-1} + a_2 S_{r+s-2} - \ldots + (-)^s a_s S_r;$$

consequently

$$H_{(t)}(r) - H_{(t-1)}(r \,.\, 1) + H_{(t-2)}(r \,.\, 1^2) - \ldots + (-)^t (r \,.\, 1^t)$$
$$= H_{(t)} S_r + H_{(t-1)} (S_{r+1} - a_1 S_r) + \ldots$$
$$+ [S_{r+t} - a_1 S_{r+t-1} + a_2 S_{r+t-2} - \ldots + (-)^t a_t S_r],$$
$$= S_{r+t} + S_{r+t-1} (H_{(1)} - a_1) + S_{r+t-2} (H_{(2)} - a_1 H_{(1)} + a_2) + \ldots$$
$$+ S_r [H_{(t)} - a_1 H_{(t-1)} + a_2 H_{(t-2)} - \ldots + (-)^t a_t];$$

therefore, by (6),

$$H_t(r) - H_{(t-1)}(r \,.\, 1) + H_{(t-2)}(r \,.\, 1^2) - \ldots + (-)^t (r \,.\, 1^t) = S_{r+t} \ldots (25).$$

Let now

$$V_{-1} = a_1 \frac{d}{da_0} + 2a_2 \frac{d}{da_1} + 3a_3 \frac{d}{da_2} + \quad \ldots\ldots\ldots\ldots\ldots(26),$$

then, from equations (7),

$$V_{-1} = a_1 d_0 - (a_1 H_{(1)} - 2a_2)\, d_1 + (a_1 H_{(2)} - 2a_2 H_{(1)} + 3a_3)\, d_2 - \ldots,$$

and therefore, from equation (24),

$$V_{-1} = S_1 d_0 - S_2 d_1 + S_3 d_2 - S_4 d_3 + \ldots + (-)^r S_{r+1} d_r + \quad \ldots\ldots(27).$$

Also let

$$V_{-r} = (r) \frac{d}{da_0} + (r\,.\,1) \frac{d}{da_1} + (r\,.\,1^2) \frac{d}{da_2} + \ldots + (r\,.\,1^s) \frac{d}{da_s} + \ldots(28);$$

then, from equations (7),

$$V_{-r} = (r)\, d_0 - \{H_{(1)}(r) - (r\,.\,1)\}\, d_1 + \{H_{(2)}(r) - H_{(1)}(r\,.\,1) + (r\,.\,1^2)\} d_2$$
$$- \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots;$$

therefore, from equation (25),

$$V_{-r} = S_r d_0 - S_{r+1} d_1 + S_{r+2} d_2 - \ldots + (-)^t S_{r+t} d_t + \ldots\ldots\ldots(29).$$

Suppose now the subject of operation to be (λ^l), we find

$$V_{-r}(\lambda^l) = \lambda \{S_r(\lambda^l) - S_{r+\lambda}(\lambda^{l-1}) + S_{r+2\lambda}(\lambda^{l-2}) - \ldots + (-)^l S_{r+l\lambda}\};$$

and since

$$S_\mu(\lambda^l) = (\lambda^l\,.\,\mu) + (\lambda^{l-1}\,.\,\lambda + \mu),$$

unless $\lambda = \mu$, when

$$S_\lambda(\lambda^l) = (l+1)\,(\lambda^{l+1}) + (\lambda^{l-1}\,.\,2\lambda),$$

we obtain, by substituting in each term of the dexter of the foregoing,

$$V_{-r}(\lambda^l) = \lambda\,(\lambda^l r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30),$$

$$V_{-\lambda}(\lambda^l) = \lambda\,(l+1)(\lambda^{l+1}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

So that V_{-r} is an effective inverse operator by means of which $(\lambda^l r)$ can be calculated by means of a single operation performed upon (λ^l).

If $V^r_{-\lambda}$ denote the performance of $V_{-\lambda}$, r times, clearly

$$V^r_{-\lambda}(\lambda^l) = \lambda^r\,(l+1)\,(l+2) \ldots (l+r)\,(\lambda^{l+r}) \quad \ldots\ldots\ldots\ldots(32).$$

It may further be shown that

$$V_{-\nu}(\lambda\,.\,\mu) = \lambda\,(\lambda\,.\,\mu\,.\,\nu) + \overline{\lambda - \mu}\,(\lambda\,.\,\mu + \nu) \quad \ldots\ldots\ldots\ldots(33),$$

$$V_{-\lambda}(\lambda\,.\,\mu) = 2\lambda\,(\lambda^2\mu) + \overline{\lambda - \mu}\,(\lambda\,.\,\mu + \lambda) \quad \ldots\ldots\ldots\ldots(34),$$

$$V_{-t}(\lambda\,.\,\mu\,.\,\nu) = \lambda\,(\lambda\,.\,\mu\,.\,\nu\,.\,t) + \overline{\lambda - \mu}\,(\lambda\,.\,\mu + t\,.\,\nu) + \overline{\lambda - \nu}\,(\lambda\,.\,\mu\,.\,\nu + t) \ldots(35),$$

$$V_{-\lambda}(\lambda\,.\,\mu\,.\,\nu) = 2\lambda(\lambda^2\,.\,\mu\,.\,\nu) + \overline{\lambda - \mu}\,(\lambda\,.\,\mu + \lambda\,.\,\nu) + \overline{\lambda - \nu}\,(\lambda\,.\,\mu\,.\,\nu + \lambda) \ldots(36);$$

for example

$$V_{-2}(5\,.\,4\,.\,3) = 5\,(5\,.\,4\,.\,3\,.\,2) + 2\,(5^2\,.\,4) + (6\,.\,5\,.\,3).$$

By means of the operator V_{-1} performed on (4^3), we find that

$$\begin{aligned}(4^31) = &-13a_{13}+9a_1a_{12}-13a_2a_{11}+13a_3a_{10}-23a_4a_9-7a_5a_8\\ &+13a_6a_7-9a_1^2a_{11}-22a_1a_2a_{10}+14a_1a_3a_9+5a_2^2a_9-6a_2a_3a_8\\ &-13a_3^2a_7+18a_1a_4a_8+10a_2a_4a_7+10a_3a_4a_6-7a_4^2a_5-2a_1a_5a_7\\ &-6a_2a_5a_6+7a_3a_5^2-11a_1a_6^2+9a_1^3a_{10}-5a_1^2a_2a_9+5a_2^2a_4a_5\\ &-9a_1^2a_3a_8+5a_1a_2^2a_8+14a_1a_2a_3a_7-a_1a_3^2a_6-5a_1a_2a_5^2-9a_1^2a_4a_7\\ &+11a_1^2a_5a_6-6a_1a_2a_4a_6-2a_1a_3a_4a_5+3a_1a_4^3+a_2^2a_3a_6-a_2a_3^2a_5\\ &-5a_2^3a_7-3a_2a_3a_4^2+a_3^3a_4.\end{aligned}$$

From §1, if in this we put $a_0 = 0$, and then decrease each suffix by unity, we obtain $(4^2 1)$, and if we put $a_0 = a_1 = 0$, and decrease each suffix by 2, we obtain (41): facts which afford a partial verification. The effect of any operator is easily examined; *ex. gr.*, consider the operator

$$\Gamma = a_1\frac{d}{da_0}+a_2\frac{d}{da_1}+a_3\frac{d}{da_2}+\ldots,$$

we have

$$\Gamma = H_{(1)}d_0-H_{(2)}d_1+H_{(3)}d_2-\ldots$$

Hence, from the relation $d_\kappa H_{(\lambda)} = (-)^{\kappa+1}H_{(\lambda-\kappa)}$,

$$\Gamma H_{(\lambda)} = \lambda H_{(1)}H_{(\lambda)}-H_{(2)}H_{(\lambda-1)}-H_{(3)}H_{(\lambda-2)}-\ldots-H_{(\lambda+1)} \quad \ldots\ldots(37),$$

and

$$\Gamma(\lambda) = \lambda\{S_1S_\lambda-H_{(\lambda+1)}\},$$

or, making use of the relation (20),

$$\Gamma(\lambda) = \lambda(\lambda 1)+\frac{\lambda}{\lambda+1}\frac{d}{da_0}S_{\lambda+1} \quad \ldots\ldots(38).$$

It should be observed that, if a symmetric function be expressed in terms of the sums of the powers of the roots, we can employ a formula given by Brioschi to throw the operator V_{-r} into another form; for Brioschi's formula is

$$\frac{d}{ds_r} = -\frac{1}{r}\left(\frac{d}{da_r}+a_1\frac{d}{da_{r+1}}+a_2\frac{d}{da_{r+2}}+\ldots\right) = -\frac{1}{r}d_r.$$

Consequently,

$$V_{-r} = S_rd_0-S_{r+1}\frac{d}{ds_1}-2S_{r+2}\frac{d}{ds_2}-\ldots-\lambda S_{r+\lambda}\frac{d}{ds_\lambda} \quad \ldots\ldots(39).$$

5. The expression $H_{(n)}$, for the sum of the homogeneous symmetric functions of weight n, will now be further examined, and it becomes necessary to amplify its definition, so that we may bring into view other like functions possessing closely analogous properties.

Definition.—Let $H_{(\lambda^l)}$ be a function such that

$$(-)^{l+1}\{H_{(\lambda)^l}-(\lambda^l)\}$$

represents the sum of all the homogeneous symmetric functions of

weight $l\lambda$ and of degree $\lambda-1$ at most. Further, let $S\,\overline{w\,.\,j}$ be the sum of all the homogeneous symmetric functions of weight w, and of degree j at most; so that

$$S\,\overline{l\lambda\,.\,\lambda-1} = (-)^{l+1}\{H_{(\lambda^l)} - (\lambda^l)\} \ldots\ldots\ldots\ldots\ldots\ldots (40).$$

The meaning of $H_{(n)}$ is clearly unaltered by the change of definition; viz., it still represents the sum of all the homogeneous symmetric functions of weight n.

It may here be remarked with regard to $H_{(n)}$, that sum of all positive coefficients = sum of all negative coefficients $= 2^{n-2}$.

It has been proved (*ante*), and reflection shows it to be obvious, that the effect of the symbol $\frac{\delta}{\Delta}$ is equivalent to another operation; that is, if the subject be ϕ, of degree λ, then

$$\frac{\delta}{\Delta}\phi = \phi - \frac{1}{\lambda}\frac{d\phi}{da_0};$$

whence

$$\frac{\delta}{\Delta}(\lambda^2) = H_{(\lambda)}(\lambda) - H_{(2\lambda)},$$

$$\frac{\delta}{\Delta}(\lambda^3) = H_{(\lambda)}(\lambda^2) - H_{(2\lambda)}(\lambda) + H_{(3\lambda)};$$

and, generally,

$$\frac{\delta}{\Delta}(\lambda^l) = H_{(\lambda)}(\lambda^{l-1}) - H_{(2\lambda)}(\lambda^{l-2}) + H_{(3\lambda)}(\lambda^{l-3}) - \ldots (-)^{l+1} H_{l\lambda} \ldots (41).$$

Now, we find by trial that

$$\frac{\delta}{\Delta}(2^2) = (2^2) - (1^4),$$

$$\frac{\delta}{\Delta}(2^3) = (2^3) + (1^6),$$

$$\frac{\delta}{\Delta}(3^2) = (3^2) - (2^3) - (2^2 1^2) - (2\,1^4) - (1^6),$$

$$\frac{\delta}{\Delta}(3^3) = (3^3) + (2^4 1) + (2^3 1^3) + (2^2 1^5) + (2\,1^7) + (1^9),$$

$$\frac{\delta}{\Delta}(4^2) = (4^2) - (3^2 2) - (3^2 1^2) - (3\,2^2 1) - (2^4) - (3\,2\,1^3) - (3\,1^5) - (2^3 1^2) - (2^2 1^4) - (2\,1^6) - (1^8),$$

and so on; so that apparently

$$(-)^{l+1}\left\{\frac{\delta}{\Delta}(\lambda^l) - (\lambda^l)\right\} = S\,\overline{l\lambda\,.\,\lambda-1},$$

or

$$H_{(\lambda^l)} = \frac{\delta}{\Delta}(\lambda^l) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42),$$

from which it appears that $H_{(\lambda^l)}$ is derived from (λ^l) in a precisely similar way to that in which $H_{(\lambda)}$ is obtained from (λ).

Also from (41) we obtain the formula

$$H_{(\lambda^l)} = H_{(\lambda)}(\lambda^{l-1}) - H_{(2\lambda)}(\lambda^{l-2}) + H_{(3\lambda)}(\lambda^{l-3}) - \ldots (-)^{l+1} H_{(l\lambda)} \quad \ldots(43).$$

Remembering the law of operation of d_λ, we have, starting from

$$S\,\overline{l\lambda \,.\, \lambda-1} = (-)^{l+1}\{H_{(\lambda^l)} - (\lambda^l)\},$$

$$S\,\overline{l\lambda-1 \,.\, \lambda-1} = (-)^{l+1} d_1 H_{(\lambda^l)} = (-)^{l+1} d_1 \frac{\delta}{\Delta}(\lambda^l),$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$S\,\overline{l\lambda-k \,.\, \lambda-1} = (-)^{l+k} d_k H_{(\lambda l)} = (-)^{l+k} d_k \frac{\delta}{\Delta}(\lambda^l) \ldots\ldots\ldots\ldots(44),$$

where $k \not> \lambda - 1$.

These equations, with those before obtained, suffice in every case to calculate $S\,\overline{w \,.\, j}$.

Now, in equation (43), we see that, on the right-hand side, each term has a factor of the form $H_{(\mu)}$; therefore, on putting all the coefficients equal to unity, the right-hand side must vanish. Consequently, as regards the algebraic sum of the coefficients, the function $H_{(\lambda^l)}$ obeys the same law as $H_{(\lambda)}$; this law may be employed in conjunction with the operation $\frac{\delta}{\Delta}$ to verify the value of (λ^l).

The preceding results lead to the general expression of symmetric function (3^κ), for

$$S\,\overline{3\kappa \,.\, 2} = (-)^{\kappa+1}\left\{\frac{\delta}{\Delta}(3^\kappa) - (3^\kappa)\right\};$$

and, since

$$\frac{\delta}{\Delta}(3^\kappa) = (3^\kappa) - \frac{1}{3}\frac{d}{da_0}(3^\kappa),$$

we have

$$S\,\overline{3\kappa \,.\, 2} = (-)^\kappa \frac{1}{3}\frac{d}{da_0}(3^\kappa),$$

and

$$(3^\kappa) = 3\,(-)^\kappa \int^{a_0} S\,\overline{3\kappa \,.\, 2}\, da_0 + ((3^{\kappa-1}));$$

the double bracket indicating as before a unit increase of suffixes.

Now, the expression of all the symmetric functions $S\,\overline{3\kappa \,.\, 2}$ has been given by Serret (*vide Algèbre Supérieure*), whence we derive at once that the coefficient of $a_s a_{3\kappa-s}$ in $S\,\overline{3\kappa \,.\, 2}$ is

$$1 - \frac{3\kappa-2s}{1!} + \frac{(3\kappa-2s)(3\kappa-2s-3)}{2!} - \frac{(3\kappa-2s)(3\kappa-2s-4)(3\kappa-2s-5)}{3!}$$
$$+ \ldots\ldots,$$

which is equal to

$$1-(3\kappa-2s)\,.\,\frac{1}{3\kappa-2s}\left\{1-2\cos\frac{(3\kappa-2s)\,\pi}{3}\right\}$$

(*vide* Todhunter's "Theory of Equations," 3rd ed., pp. 186, 187), and this is

$$2\cos\frac{3\kappa-2s}{3}\pi.$$

There are two obvious exceptions, viz., when

$$3\kappa-2s \text{ is either zero or } \pm 1,$$

and then the coefficient is unity.

Therefore, substituting and performing the integration, we arrive at finally the following result :—

$$(3^\kappa) = 3!\,(-)^\kappa\left[\frac{1}{2!}\cos\kappa\pi\,.\,a_{3\kappa}+\sum_{s<\frac{1}{2}(3\kappa-1)}^{s=1}\cos\frac{(3\kappa-2s)\,\pi}{3}\,a_s\,a_{3\kappa-s}\right.$$
$$\left.+\frac{1}{2}\sum^{s\gtreqless\frac{1}{2}(3\kappa-1)}\{a_s\,a_{3\kappa-s}\}\right]+((3^{\kappa-1})).$$

Ex. gr., $(3^5) = 3a_{15}-3a_1a_{14}-3a_2a_{13}+6a_3a_{12}-3a_4a_{11}-3a_5a_{10}$
$+6a_6a_9-3a_7a_8+((3^4))$.

This formula gives an important series of perpetuants.

6. It was proved in § 4 that

$$H_{(t)}(r)-H_{(t-1)}(r1)+H_{(t-2)}(r1^2)-\ldots+(-)^t(r1^t)=S_{r+t},$$

with a modification when $r=1$.

The object here is to examine in some cases the expression

$$H_{(t)}(\lambda\,.\,\mu\,.\,\nu\ldots)-H_{(t-1)}(\lambda\,.\,\mu\,.\,\nu\ldots 1)+H_{(t-2)}(\lambda\,.\,\mu\,.\,\nu\ldots 1^2)-\ldots$$
$$\ldots+(-)^s H_{(t-s)}(\lambda\,.\,\mu\,.\,\nu\ldots 1^s)+\ldots+(-)^t(\lambda\,.\,\mu\,.\,\nu\ldots 1^t),$$

which for brevity will be denoted by

$$F\{(t)\,.\,(\lambda\,.\,\mu\,.\,\nu\ldots)\}.$$

Firstly, F is in every case a non-unitary symmetric function, since the operation d_1 clearly makes it vanish.

Let $T(a, b, c, d)$ represent the sum of the non-unitary symmetric functions of weight a, of b parts, and having no part less than c or greater than d; then the following relations may be verified, viz. :—

$$F\{(0)\,.\,(2^2)\} = \quad (2^2) \quad = T(4\,.\,2\,.\,2\,.\,2+0),$$
$$F\{(1)\,.\,(2^2)\} = \quad (32) \quad = T(5\,.\,2\,.\,2\,.\,2+1),$$
$$F\{(2)\,.\,(2^2)\} = (42)+(3^2) = T(6\,.\,2\,.\,2\,.\,2+2),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$F\{(t)\,.\,(2^2)\} = T(t+4\,.\,2\,.\,2\,.\,2+t) \ldots\ldots\ldots\ldots(45),$$

and $$F\{(t).(3^2)\} = T(t+6.2.3.3+t) \quad \ldots\ldots\ldots\ldots(46),$$
$$F\{(t).(4^2)\} = T(t+8.2.4.4+t) \quad \ldots\ldots\ldots\ldots(47),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and, generally, $F\{(t).(\lambda^2)\} = T(t+2\lambda.2.\lambda.\lambda+t)\ldots\ldots\ldots\ldots(48).$

These, of course, lead to a series of inverse operators as before; thus

$$(\lambda^2)\frac{d}{da_0}+(\lambda^2 1)\frac{d}{da_1}+(\lambda^2 1^2)\frac{d}{da_2}+\ldots$$
$$=(\lambda^2)d_0-F\{(1).(\lambda^2)\}d_1+F\{(2).(\lambda^2)\}d_2-\ldots+(-)^t F\{(t).(\lambda^2)\}d_t\ldots;$$

but, as the results do not appear to be very useful, no further mention will be made of them.

Proceeding, we find

$$F\{(t).(2^3)\} = T(t+6.3.2.2+t) \quad \ldots\ldots\ldots\ldots(49),$$
$$F\{(t).(3^3)\} = T(t+9.3.3.3+t) \quad \ldots\ldots\ldots\ldots(50),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$F\{(t).(\lambda^3)\} = T(t+3\lambda.3.\lambda.\lambda+t) \quad \ldots\ldots\ldots\ldots(51),$$

and, generally, $F\{(t).(\lambda^l)\} = T(t+l\lambda.l.\lambda.\lambda+t) \quad \ldots\ldots\ldots\ldots(52).$

An expression can now be obtained for the sum of all the non-unitary symmetric functions of a given weight; this in our notation is represented by (the weight being w)

$$T(w.1.2.w)+T(w.2.2.w-2)+T(w.3.2.w-4)$$
$$+T(w.4.2.w-6)+\ldots$$
$$= F\{(w-2).(2)\}+F\{(w-4).(2^2)\}+F\{(w-6).(2^3)\}$$
$$+F\{(w-8).(2^4)\}+\ldots$$
$$= H_{(w-2)}(2)-H_{(w-3)}(2\,1)+H_{(w-4)}\{(2\,1^2)+(2^2)\}$$
$$-H_{(w-5)}\{(2\,1^3)+(2^2 1)\}+H_{(w-6)}\{(2\,1^4)+(2^2\,1^2)+(2^3)\}$$
$$-\ldots+(-)^s H_{(w-s)}\{(2\,1^{s-2})+(2^2 1^{s-4})+(2^3 1^{s-6})+\ldots+(2^{\frac{1}{2}s})\text{ or }(2^{\frac{1}{2}(s-1)}\,1)\}$$
$$+\ldots\ldots \quad \ldots\ldots\ldots\ldots(53),$$

the last term being that containing $H_0 = 1$ as a factor.

Ex. gr.—When $w = 6$,

$$H_{(4)}(2)-H_{(3)}(2\,1)+H_{(2)}\{(2\,1^2)+(2^2)\}-H_1\{(2\,1^3)+(2^2 1)\}$$
$$+(2\,1^4)+(2^2 1^2)+(2^3)$$
$$=a_1^6-6a_1^4a_2+10a_1^2a_2^2-3a_2^3+4a_1^3a_3-11a_1a_2a_3+4a_3^2-a_1^2a_4+3a_2a_4-a_1a_5+a_6$$
$$=(6)+(42)+(3^2)+(2^3).$$

An expression can also be deduced for the sum of the non-unitary symmetric functions of weight w, which contain no part less than j;

for, in the notation of this paper, this is

$$\begin{aligned}
&T(w\,.\,1\,.\,j\,.\,w)+T(w\,.\,2\,.\,j\,.\,w-j)+T(w\,.\,3\,.\,j\,.\,w-2j)\\
&\qquad\qquad +T(w\,.\,4\,.\,j\,.\,w-3j)+\ldots\\
=\;& F\{(w-j)\,.\,(j)\}+F\{(w-2j)\,.\,(j^2)\}+F\{(w-3j)\,.\,(j^3)\}+\ldots\\
=\;& H_{(w-j)}(j)-H_{(w-j-1)}(j1)+H_{(w-j-2)}(j1^2)-\ldots\\
&+H_{(w-2j)}\{(j^2)+(-)^j(j1^j)\}-H_{(w-2j-1)}\{(j^21)+(-)^j(j1^{j+1})\}\\
&\qquad +H_{(w-2j-2)}\{(j^21^2)+(-)^j(j1^{j+2})\}-\ldots\\
&+H_{(w-3j)}\{(j^3)+(-)^{2j}(j1^{2j})\}-H_{(w-3j-1)}\{(j^31)+(-)^{2j}(j1^{2j+1})\}\\
&\qquad +H_{(w-3j-2)}\{(j^31^2)+(-)^{2j}(j1^{2j+2})\}-\ldots\\
&+\ldots\ldots\\
&+H_{(w-ij)}\{(j^i)+(-)^{ij-j}(j1^{ij-j})\}-H_{(w-ij-1)}\{(j^i1)+(-)^{ij-j}(j1^{ij-j+1})\}\\
&\qquad +H_{(w-ij-2)}\{(j^i1^2)+(-)^{ij-j}(j1^{ij-j+2})\}-\ldots\\
&+\ldots\ldots \qquad \ldots\ldots\ldots\

the terms being continued until the first part of the partition is about to become less than the second part.

Analogous results are obtained in the case of the symmetric functions of a greater number of parts, but it does not seem worth while presenting them.

7. Newton's formula for the sums of the powers of the roots,

$$S_w - a_1 S_{w-1} + a_2 S_{w-2} - \ldots (-)^{w+1} w a_w = 0,$$

admits of generalisation.

For this purpose, S_w must not be regarded as essentially the sum of the w^{th} powers of the roots, but simply as the sum of the one-part partition symmetric functions of weight w; the notation S^r_w is then adopted to signify the sum of the r-part partition symmetric functions of weight w, and what was before denoted by S_w becomes in the new notation S^1_w.

Let now the r^{th} number in the series of polygonal numbers of the n^{th} order be denoted by n_r, so that the series of numbers of the 1st, 2nd, 3rd, &c. orders are, written horizontally,

$$\begin{array}{cccccc} 1_1, & 1_2, & 1_3, & 1_4 \ldots\ldots \\ 2_1, & 2_2, & 2_3, & 2_4 \ldots\ldots \\ 3_1, & 3_2, & 3_3, & 3_4 \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots ; \end{array}$$

then we have the following system of equations, viz.,—

$$\begin{aligned}
S^0_w &= 1_1 a_0 H_{(w)} - 1_2 a_1 H_{(w-1)} + 1_3 a_2 H_{(w-2)} - \ldots (-)^w 1_{w+1} a_w, \\
S^1_w &= 2_1 a_1 H_{(w-1)} - 2_2 a_2 H_{(w-2)} + 2_3 a_3 H_{(w-3)} - \ldots (-)^{w+1} 2_w a_w, \\
S^2_w &= 3_1 a_2 H_{(w-2)} - 3_2 a_3 H_{(w-3)} + 3_3 a_4 H_{(w-4)} - \ldots (-)^{w+2} 3_{w-1} a_w, \\
S^3_w &= 4_1 a_3 H_{(w-3)} - 4_2 a_4 H_{(w-4)} + 4_3 a_5 H_{(w-5)} - \ldots (-)^{w+3} 4_{w-2} a_w, \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
S^r_w &= \overline{r+1}_1 a_r H_{(w-r)} - \overline{r+1}_2 a_{r+1} H_{(w-r-1)} + \overline{r+1}_3 a_{r+2} H_{(w-r-2)} - \ldots \\
&\qquad \ldots + (-)^{w+r} \overline{r+1}_{w-r+1} a_w \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56).
\end{aligned}$$

These equations may be all readily verified by means of Mr. Hammond's operators; from the general term in $H_{(w)}$, S^r_w can therefore be easily calculated.

Also, starting with Newton's formula,

$$S^1_w - a_1 S^1_{w-1} + a_2 S^1_{w-2} - a_3 S^1_{w-3} + \ldots + (-)^{w+1} 2_w a_w = 0,$$

we deduce

$$\begin{aligned}
S^2_w - a_1 S^2_{w-1} + a_2 S^2_{w-2} - a_3 S^2_{w-3} + \ldots + (-)^{w+1} 3_{w-1} a_w &= 0, \\
S^3_w - a_1 S^3_{w-1} + a_2 S^3_{w-2} - a_3 S^3_{w-3} + \ldots + (-)^{w+1} 4_{w-2} a_w &= 0,
\end{aligned}$$

and generally

$$S^r_w - a_1 S^r_{w-1} + a_2 S^r_{w-2} - a_3 S^r_{w-3} + \ldots + (-)^{w+1} \overline{r+1}_{w-r+1} a_w = 0 ;$$

or, putting for the polygonal number of the $\overline{r+1}^{\text{th}}$ order its value $\frac{w\,!}{r\,!\,(w-r)\,!}$, the formula is perhaps better written in the form

$$S^r_w - a_1 S^r_{w-1} + a_2 S^r_{w-2} - \ldots + (-)^{w+1} \frac{w\,!}{r\,!\,(w-r)\,!} a_w = 0 \ldots\ldots (57),$$

which is the generalisation of Newton's formula referred to at the commencement of this section. The function S^r_w further possesses other properties, analogous and very similar to those possessed by S^1_w, and in all cases the generalisation is effected by means of the polygonal numbers. To obtain the general term in S^r_w it is convenient to extend the notion of the weight of a term, as defined in Salmon's *Higher Algebra;* for this purpose consider any term

$$a_1^{\alpha_1} a_2^{\alpha_2} \ldots\ldots\ldots a_r^{\alpha_r} a_{r+1}^{\alpha_{r+1}} \ldots\ldots\ldots ;$$

and define the r-gonal weight of this term to be

$$w_r = r_1 \alpha_{r-1} + r_2 \alpha_r + r_3 \alpha_{r+1} + r_4 \alpha_{r+2} + \ldots\ldots\ldots\ldots\ldots (58),$$

wherein $r_1, r_2, r_3 \ldots$ represent the successive r-gonal numbers; then what is ordinarily denoted by the weight will be the weight of the second order, viz.—

$$w_2 = 2_1 \alpha_1 + 2_2 \alpha_2 + 2_3 \alpha_3 + 2_4 \alpha_4 + \ldots\ldots ;$$

then the theorem is that

$$S^r_w = \Sigma\, (-)^{w_2+k+r-1} \frac{(k-1)\,!\; w_{r+1}}{\alpha_1\,!\;\alpha_2\,!\;\alpha_3\,!\ldots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \ldots\ldots\ldots\ldots\ldots (59),$$

wherein $$\alpha_1 + \alpha_2 + \alpha_3 + \ldots = k.$$

When $r = 1$, this becomes Waring's formula for the sums of the powers of the roots, and the general theorem is proved as follows:—

Since $$S^r_w = \overline{r+1}_1 a_r H_{(w-r)} - \overline{r+1}_2 a_{r+1} H_{(w-r-1)} + \ldots + (-)^{w+r} \overline{r+1}_{w-r+1} a_w,$$

and $$H_{(w)} = \Sigma\, (-)^{w+k} \frac{k\,!}{\alpha_1\,!\;\alpha_2\,!\;\alpha_3\,!\ldots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \ldots,$$

we obtain, without difficulty,

$$S^r_w = \Sigma\, (-)^{w_2+k+r-1} \frac{(k-1)\,!\;\{\overline{r+1}_1 a_r + \overline{r+1}_2 a_{r+1} + \ldots\}}{\alpha_1\,!\;\alpha_2\,!\;\alpha_3\,!} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3}$$

$$= \Sigma\, (-)^{w_2+k+r-1} \frac{(k-1)\,!\; w_{r+1}}{\alpha_1\,!\;\alpha_2\,!\;\alpha_3\,!\ldots} a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} \ldots$$

This theorem is extremely useful in the verification of symmetric functions, and Professor Cayley's law of symmetry enables it to answer a two-fold purpose, viz.,—firstly, as a summation theorem for the r-part partition symmetric functions of weight w; and, secondly, as a theorem for the algebraic sum of the numerical coefficients of the

r-part partition terms, in the expression in terms of the literal coefficients of all symmetric functions of weight w. For instance, the algebraic sum of the numerical coefficients of the r-part partition terms in symmetric function

$$(\lambda^l . \mu^m . \nu^n \ldots)$$

is
$$(-)^{w+l+m+n+\ldots+r-1} \frac{(l+m+n+\ldots-1)!\, w_{r+1}}{l!\, m!\, n! \ldots} \quad \ldots\ldots\ldots\ldots(60),$$

wherein w_{r+1} is the $r+1$-gonal weight of the partition; so that, in calculating a symmetric function independently, by means of the differential equation, we can write down at once relations that must be satisfied by the coefficients, equal in number to the highest index occurring in the expression in the roots.

The algebraic sum of all the coefficients is obviously

$$(-)^{w+l+m+n+\ldots} \frac{(l+m+n+\ldots)!}{l!\, m!\, n! \ldots},$$

so that, as regards a symmetric function of degree p in the coefficients, we have p independent verifications of the numerical coefficients.

The general expression of S^r_n may be thrown into the determinant form, for we have the system of equations

$$\begin{aligned}
&S^r_r + (-)^{r+1}\overline{r+1}_1 a_r = 0,\\
&S^r_{r+1} - a_1 S^r_r + (-)^r \overline{r+1}_2 a_{r+1} = 0,\\
&S^r_{r+2} - a_1 S^r_{r+1} + a_2 S^r_r + (-)^{r+1}\overline{r+1}_3 a_{r+2} = 0,\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots
\end{aligned}$$

whence, solving for S^r_n, we find if r be odd,

$$S^r_n = \begin{vmatrix}
a_1 & 1 & \ldots & & & & \\
a_2 & a_1 & 1 & \ldots & & & \\
a_3 & a_2 & a_1 & 1 & \ldots & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\
\overline{r+1}_{n-r+1} a_n, & \overline{r+1}_{n-r} a_{n-1}, & \ldots \text{ \&c. } \ldots & & & & \overline{r+1}_1 a_r
\end{vmatrix} \quad \ldots\ldots\ldots\ldots(61);$$

if r be even,

$$S^r_n = \begin{vmatrix}
a_1 & -1 & & & & & \\
-a_2 & +a_1 & -1 & & & & \\
+a_3 & -a_2 & +a_1 & -1 & & & \\
\ldots & \ldots & \ldots & \ldots & & & \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & -1 \\
(-)^n \overline{r+1}_{n-r+1} a_n, & (-)^{n+1} \overline{r+1}_{n-r}, a_{n-1} \ldots \text{ \&c. } \ldots & & & & & +\overline{r+1}_1 a_r
\end{vmatrix}$$
$$\ldots\ldots\ldots\ldots(62).$$

In the annexed table are given the expressions of the single partition seminvariants, otherwise non-unitary symmetric functions of weight 13; they are intended as a contribution to the study of the

weight 13 sources of covariants of binary quantics; they were calculated as follows:—By the method of §2 the non-unitary portions of all the symmetric functions of weight 13 were calculated and tabulated vertically, headed by the partition of the function, the left-hand column representing the partitions of the terms; then of course, regarding the vertical partition column as pertaining to the functions and the horizontal partition row to the terms, the result is a table of the non-unitary symmetric functions.

As every possible verification afforded by the general expression for S^r_{13} was employed, it is scarcely possible that there should be an error; *ex. gr.*, consider the ten-part partition terms: we find

$$\begin{aligned} S^{10}_{13} &= 11_1 a_{10} H_{(3)} - 11_2 a_{11} H_{(2)} + 11_3 a_{12} H_{(1)} - 11_4 a_{13}, \\ &= a_{10} \cdot a_3 - 11 a_{11}(-a_2) - \frac{13!}{10!\,3!} a_{13} + \text{terms containing } a_1, \\ &= a_3 a_{10} + 11 a_2 a_{11} - 286 a_{13} + \ldots\ldots, \end{aligned}$$

whilst the columns headed by partitions of ten parts are

	(4.1^9)	$(3.2.1^8)$	$(2^3.1^7)$		S^{10}_{13}
a_{13}	-13	-117	-156	$=$	-286
$a_2 a_{11}$	$+2$	$+18$	-9	$=$	$+11$
$a_3 a_{10}$	$+3$	-3	$+1$	$=$	$+1$

which is correct, and is given as a specimen of the verifications employed throughout.

The terms are arranged on Mr. Durfee's plan. The general values of S^2_n, S^3_n, ... for a few terms are:—

$$\begin{aligned} S^2_n = {} & a_1^{n-2} a_2 - 3a_1^{n-3} a_3 + a_1^{n-4}\{6a_4 - (n-3)\,a_2^2\} \\ & - a_1^{n-5}\{10a_5 - (1+3)\,(n-4)\,a_2 a_3\} \\ & + a_1^{n-6}\left[15a_6 - (n-5)\{(1+6)\,a_2 a_4 + 3a_3^2\} + (n-4)(n-5)\frac{a_2^3}{2!}\right] \\ & - a_1^{n-7}\left[21a_7 - (n-6)\,\{(1+10)\,a_2 a_5 + (3+6)\,a_3 a_4\}\right. \\ & \qquad \left. + (2.1+3)(n-5)(n-6)\frac{a_2^2 a_3}{2!}\right] \\ & + a_1^{n-8}\left[28a_8 - (n-7)\{(1+15)\,a_2 a_6 + (3+10)\,a_3 a_5 + 6a_4^2\}\right. \\ & \qquad + (n-6)(n-7)\left\{(2.1+6)\frac{a_2^2 a_4}{2!} + (1+2.3)\frac{a_2 a_3^2}{2!}\right\} \\ & \qquad \left. - (n-5)(n-6)(n-7)\frac{a_2^4}{3!}\right] \\ & - \ldots\ldots\ldots\ldots \end{aligned}$$

$$S_n^3 = a_1^{n-3}a_3 - 4a_1^{n-4}a_4 + a_1^{n-5}\{10a_5 - (n-4)\,a_2a_3\}$$
$$- a_1^{n-6}[20a_6 - (n-5)\{4a_2a_4 + a_3^2\}]$$
$$+ a_1^{n-7}\left[35a_7 - (n-6)\{10a_2a_5 + (1+4)a_3a_4\} + (n-5)(n-6)\frac{a_2^2a_3}{2!}\right]$$
$$- a_1^{n-8}\left[56a_8 - (n-7)\{20a_2a_6 + (1+10)a_3a_5 + 4a_4^2\}\right.$$
$$\left. + (n-6)(n-7)\left\{a_2a_3^2 + 4\frac{a_2^2a_4}{2!}\right\}\right]$$
$$+ \ldots\ldots\ldots\ldots\ldots$$

$$S_n^4 = a_1^{n-4}a_4 - 5a_1^{n-5}a_5 + a_1^{n-6}\{15a_6 - (n-5)\,a_2a_4\}$$
$$- a_1^{n-7}[35a_7 - (n-6)\{5a_2a_5 + a_3a_4\}]$$
$$+ a_1^{n-8}\left[70a_8 - (n-7)\{15a_2a_6 + 5a_3a_5 + a_4^2\} + (n-6)(n-7)\frac{a_2^2a_4}{2!}\right]$$
$$- a_1^{n-9}\left[126a_9 - (n-8)\{35a_2a_7 + 15a_3a_6 + (1+5)\,a_4a_5\}\right.$$
$$\left. + \ldots\ldots\ldots\ldots, \qquad + (n-7)(n-8)\left\{5\frac{a_2^2a_5}{2!} + a_2a_3a_4\right\}\right]$$

and so on.

Another means of verifying any symmetric function may here be noted. Consider the equation

$$x^n + \frac{x^{n-1}}{1!} + \frac{x^{n-2}}{2!} + \frac{x^{n-3}}{3!} + \ldots + \frac{1}{n!} = 0,$$

or say $\phi(x) = 0$; we have

$$\frac{\phi(x)}{x^n} = e^{+\frac{1}{x}} - \frac{x^{-n-1}}{(n+1)!} - \frac{x^{-n-2}}{(n+2)!} - \ldots$$
$$= e^{+\frac{1}{x}}\left\{1 - \frac{e^{-\frac{1}{x}}x^{-n-1}}{(n+1)!} - \frac{e^{-\frac{1}{x}}x^{-n-2}}{(n+2)!} - \ldots\right\},$$

$$\therefore \log\frac{\phi(x)}{x^n} = +\frac{1}{x} + \log\left\{1 - \frac{e^{-\frac{1}{x}}x^{-n-1}}{(n+1)!} - \frac{e^{-\frac{1}{x}}x^{-n-2}}{(n+2)!} - \ldots\right\}$$
$$= +\frac{1}{x} + \text{terms involving } \frac{1}{x^{n+1}} \text{ and higher powers of } \frac{1}{x},$$

but (Todhunter's *Theory of Equations*, p. 181)

$-\frac{1}{m}S_m^1$ is the coefficient of $\frac{1}{x^m}$ in the dexter of this identity.

Consequently the sums of all powers of the roots of the above equation from the 2nd to the n^{th} inclusive vanish, and therefore also every symmetric function of the roots to the n^{th} degree inclusive (with the exception of the coefficients of the equation) vanishes. Therefore the expression of a symmetric function vanishes on putting for each literal coefficient the reciprocal of the factorial of its suffix.

[I ought to make clear my acknowledgments to Mr. Hammond for many suggestions on this paper which I have almost invariably adopted.]

SINGLE PARTITION SEMINVARIANTS, OTHERWISE NON-UNITARY SYMMETRIC FUNCTIONS OF WEIGHT 13.

	(13)	(12 . 1)	(11 . 2)	(10 . 3)	(9 . 4)	(8 . 5)	(7 . 6)	(11 . 1²)	(10 . 2 . 1)	(9 . 3 . 1)	(9 . 2²)	(8 . 4 . 1)	(8 . 3 . 2)	(7 . 5 . 1)	(7 . 4 . 2)	(7 . 3²)	(6 . 4 . 3)
(13)	+13	−13	−13	−13	−13	−13	−13	+13	+26	+26	+13	+26	+26	+26	+26	+13	+26
(11 . 2)	−13	+13	−9	+13	+13	+13	+13	−2	−4	−26	+9	−26	−4	−26	−4	−13	−26
(10 . 3)	−13	+13	+13	−17	+13	+13	+13	+13	+4	+4	−13	−26	+4	−26	−26	−17	+4
(9 . 4)	−13	+13	+13	+13	−23	+13	+13	−13	−26	+10	+5	+10	−26	+10	+10	−13	+10
(8 . 5)	−13	+13	+13	+13	+13	−27	+13	−13	−26	−26	−13	+14	+14	+14	−26	−13	−26
(7 . 6)	−13	+13	+13	+13	+13	+13	−29	−13	−26	−26	−13	−26	−26	+16	+16	+8	+16
(9 . 2²)	+13	−13	+9	−13	+5	−13	−13	+2	+4	+8	0	+8	+4	+26	−14	+13	+8
(8 . 3 . 2)	+26	−26	−4	+4	−26	+14	−26	+15	0	+22	+4	+12	+8	+12	+30	+20	+22
(7 . 4 . 2)	+26	−26	−4	−26	+10	−26	+16	+15	+30	+16	−14	+16	+30	+10	+8	−16	−26
(7 . 3²)	+13	−13	−13	+17	−13	−13	+8	+13	−4	−4	+13	+26	−4	+5	+5	−8	−25
(6 . 5 . 2)	+26	−26	−4	−26	−26	+14	+16	+15	+30	+52	+4	+12	−10	−30	−12	−16	+10
(6 . 4 . 3)	+26	−26	−26	+4	+10	−26	+16	+26	+22	−14	+8	+16	+22	+10	−26	−16	+16
(5² . 3)	+13	−13	−13	+2	−13	+27	−13	+13	+11	+11	+13	−14	−29	−14	+26	+13	+11
(5 . 4²)	+13	−13	−13	−13	+23	+7	−13	+13	+26	−10	−5	−30	+6	+6	−10	+13	−10
(7 . 2³)	−13	+13	−9	+13	−5	+13	−1	−2	−4	−8	0	−8	−4	−12	0	−6	+6
(6 . 3 . 2²)	−39	+39	−5	+9	+21	−1	−3	−17	−4	−30	−4	−20	−12	+4	+26	+12	+24
(5 . 4 . 2²)	−39	+39	−5	+39	−15	−1	−3	−17	−34	−24	+14	+16	+6	+4	+6	−18	+18
(5 . 3² . 2)	−39	+39	+17	−21	+39	−41	+18	−28	+4	−18	−17	+2	+36	+23	−35	0	+3
(4² . 3 . 2)	−39	+39	+17	+9	−33	+19	−3	−28	−26	+24	+19	+14	−30	−16	+2	+12	−6
(5 . 2⁴)	+13	−13	+9	−13	+5	−3	+1	+2	+4	+8	0	−2	−6	+2	0	+6	−6
(4 . 3³)	−13	+13	+13	−17	+1	+13	−8	−13	+4	+16	−7	−14	+4	−5	+7	−4	+1
(4 . 3 . 2³)	+52	−52	+14	−22	+20	−12	+4	+19	+8	+2	−14	−8	+16	+8	−6	−6	+6
(3³ . 2³)	+26	−26	−4	+19	−20	+14	−5	−15	+15	+1	+7	+6	−7	−9	+3	+2	+2
(3 . 2⁵)	−13	+13	−9	+7	−5	+3	−1	−2	+2	−2	0	+2	0	−2	0	0	−1

	$(5^2.3)$	(5.4^2)	(10.1^3)	$(9.2.1^2)$	$(8.3.1^2)$	$(8.2^2.1)$	$(7.4.1^2)$	$(7.3.2.1)$	(7.2^3)	$(6.5.1^2)$	$(6.4.2.1)$	$(6.3^2.1)$	$(6.3.2^2)$	$(6^2.2.1)$	$(5.4.3.1)$	$(5.4.2^2)$	$(5.3^2.2)$	$(4^2.3.2)$	$(4.3^2.1)$	(9.1^4)	$(8.2.1^3)$	$(7.2^2.1^2)$
(13)	+13	+13	−13	−39	−39	−39	−39	−78	−13	−39	−78	−39	−39	−39	−78	−39	−39	−39	−13	+13	+52	+78
(11.2)	−13	−13	+2	+6	+28	−5	+28	+34	−9	+28	+34	+39	−5	+17	+78	−5	+17	+13	+13	−2	−8	−1
(10.3)	+2	−13	+3	+9	+9	+9	+39	−12	+13	+39	+48	−21	+9	+24	+18	+39	−21	+13	−3	−12	−12	−18
(9.4)	−13	+23	+13	+3	+3	+21	+3	+42	−5	+39	+6	+3	+21	+39	−30	−15	+39	−23	−4	−16	−16	−24
(8.5)	+27	+7	+13	+39	−1	−1	−1	+38	+13	−1	+38	+39	−1	−41	−42	−1	−41	−7	−13	−12	−12	−38
(7.6)	−13	−13	+13	+39	+39	+39	−3	−6	−1	−45	−48	−24	−3	−3	+36	−3	+18	−13	−13	−52	−52	−15
(9.2^2)	+13	−5	−2	−6	−10	−4	−10	−16	0	−28	+2	−21	−4	−17	−24	+14	−17	+5	+2	+8	+8	+10
$(8.3.2)$	−29	+6	−5	−15	−21	−12	−27	−30	−4	−27	−42	−18	−12	+39	+24	+6	−35	−6	+5	+20	+20	+27
$(7.4.2)$	+26	−10	−15	−9	−31	−16	−17	−48	0	+17	+30	+21	+26	−14	−6	+6	−35	+10	+6	+24	+24	+32
(7.3^2)	−2	+13	−3	−9	−9	−9	−18	−9	−6	+3	+15	+21	+12	−3	+3	−18	0	−13	+3	+12	+12	+18
$(6.5.2)$	−14	+6	−15	−45	−27	+6	+15	+12	+10	+27	−6	−15	−12	+6	+6	−12	+15	−6	+15	+20	+20	−24
$(6.4.3)$	+11	−10	−16	−12	−12	−30	0	+54	+6	+6	0	+9	−24	−21	−18	+18	+3	+10	+7	+28	+28	−21
$(5^2.3)$	+3	−7	−8	−24	+16	+16	+1	+7	−13	+1	−23	−9	+16	+11	−3	+1	−4	+7	+8	−8	−8	+8
(5.4^2)	−7	+7	−13	−3	+17	−1	+17	−22	+5	−19	+14	−3	−1	+1	+10	−5	+1	−7	+4	−4	−4	+4
(7.2^3)	−13	+5	+2	+6	+10	+4	+10	+16	0	0	−16	0	−10	+3	+10	0	+10	−5	−2	−8	−8	−10
$(6.3.2^2)$	+16	−1	+7	+21	+31	+16	−5	−38	−10	+1	+10	+12	+16	−4	−6	−2	+10	+1	−7	−28	−28	+26
$(5.4.2^2)$	+1	−5	+17	+15	+1	−20	−13	+24	0	+1	−12	0	−2	+11	−10	0	+2	+5	−8	+8	+8	−2
$(5.3^2.2)$	−4	+1	+8	+24	−10	−19	+5	−1	+10	−1	+17	−3	−10	−11	+3	+2	+4	−1	−8	+8	+8	−5
$(4^2.3.2)$	−4	+3	+18	−18	−4	+11	0	+4	−5	+2	−2	−6	+5	−2	+8	−1	−2	−3	0	0	0	0
(5.2^4)	+3	0	−2	−6	0	+6	0	−6	0	0	+6	0	0	−3	0	0	0	0	+2	−2	−2	0
(4.3^3)	+2	−1	+3	−3	−3	+3	+6	−3	0	−3	−3	+3	0	+3	−3	0	0	+1				
$(4.3.2^3)$	−3	0	−9	+9	−5	−2	+3	+2	0	−1	−2	0	0	+1								
$(3^3.2^2)$	+1																					

	(7 . 3 . 1³)	(6 . 4 . 1³)	(6 . 3 . 2 1²)	(6 . 2³ . 1)	(6² . 1³)	(6 . 4 . 2 . 1²)	(8 . 1⁵)	(7 . 2 . 1⁴)	(6 . 3 . 1⁴)	(4² . 3 . 2)	(5 . 3² . 1²)	(7 . 1⁶)	(6 . 2² . 1³)	(6 . 2 . 1⁵)	(6 . 1⁷)	(5 . 3 . 2² . 1)	(5 . 2⁴)	(5 . 4 . 1⁴)	(5 . 3 . 2 . 1³)	(5 . 2³ . 1²)	(5 . 2² . 1⁴)
(13)	+52	+52	+156	+52	+26	+156	−13	−65	−65	−39	+78	+13	−130	+78	−13	+156	+13	−65	−260	−130	+195
(11 . 2)	−30	−30	−68	+14	−15	−68	+2	+10	+32	+17	−67	−2	+9	−12	+2	−24	+9	+32	+106	−13	−19
(10 . 3)	−12	−42	−6	−22	−21	−96	+3	+15	+15	+9	+12	−3	+30	−18	+3	−6	−13	+45	+30	+40	−45
(9 . 4)	−16	−16	−48	−16	−26	−12	+4	+20	+20	−33	−6	−4	+40	−24	+4	−84	+5	+20	+80	+40	−60
(8 . 5)	−12	−12	−76	−12	+14	+44	+5	+25	+25	+19	+22	−5	+50	−30	+5	+84	−3	−15	−60	−30	+25
(7 . 6)	−10	+32	+54	+4	+16	+12	+13	+23	−19	−3	−15	−6	−17	+6	0	−30	+1	−19	+8	+11	−6
(9 . 2²)	+12	+12	+32	+4	+15	+14	−2	−10	−14	+19	+31	+2	−18	+12	−2	+24	0	−14	−52	−14	+28
(8 . 3 . 2)	+26	+32	+66	+16	−4	−36	−5	−25	−31	−30	−21	+5	−47	+30	−5	−66	−6	+3	+48	+37	−28
(7 . 4 . 2)	+32	−10	−38	−26	−1	−4	−6	−30	+4	+2	+10	+6	+28	−6	0	+38	0	+4	−18	−6	+2
(7 . 3²)	+12	0	−36	−6	0	+12	−3	−15	+6	+12	−12	+3	+12	−3	0	+6	+6	−3	+12	−12	+3
(6 . 5 . 2)	0	−12	+24	+2	−11	+6	−7	+7	−3	+6	−3	0	−2	0	0	−6	0	+7	−4	+2	0
(6 . 4 . 3)	−14	−2	−12	+18	+5	+12	−7	+7	+7	−6	+3	0	−7	0	0	0	−6	−5	−2	+3	0
(5² . 3)	−8	+7	+1	−3	−4	+1	0	0	0	−4	+8	0	0	0	0	−9	+3	0	0	0	0
(5 . 4²)	−4	−4	+8	−4	+6	−8	0	0	0	+3	−4	0	0	0	0	+4	0	0	0	0	0
(7 . 2³)	−12	+2	+10	+10	−1	0	+2	+10	0	−5	−10	−2	−10	+2	0	−10	0	0	+10	0	0
(6 . 3 . 2²)	+4	−2	−14	−6	+1	+4	+7	−7	+3	+5	+2	0	+2	0	0	+6	0	−1	−2	0	0
(5 . 4 . 2²)	−4	+8	−4	+2	−4	0	0	0	0	−1	+4	0	0	0	0	−2	0	0	0	0	0
(5 . 3² . 2)	+2	−7	+5	0	+4	−1	0	0	0	−2	−2	0	0	0	0	0	0	0	0	0	0
(4² . 3 . 2)	0	0	0	0	0	0	0	0	0	+1											
(5 . 2⁴)	+2	−2	0	0	+1																

	$(5^3.1^5)$	$(5^2.1^6)$	(5.1^8)	(4.3^3)	$(4^2.3.1^2)$	$(4^2.2^2.1)$	$(4.3^2.2.1)$	$(4.3^2.1^3)$	$(4^2.2.1^3)$	$(4.3^2.1^3)$	$(4.3^2.2^2.1^2)$	$(4.2^4.1)$	$(4^2.1^5)$	$(4.3.2.1^4)$	$(4.2^3.1^3)$	$(4.3.1^6)$	$(4.2^2.1^5)$	$(4.2.1^7)$	(4.1^9)	$(3^4.1)$
(13)	+78	−91	+13	−13	+78	+78	+156	+52	−130	−130	−390	−65	+39	+390	+260	−91	−273	+104	−13	+13
(11.2)	−34	+14	−2	+13	−67	−12	−90	+14	+53	+97	+93	−23	−17	−148	+4	+36	+31	−16	+2	−13
(10.3)	−18	+21	−3	−17	−33	−48	+54	−22	+75	0	+30	+35	−24	−60	−70	+21	+63	−24	+3	+17
(9.4^1)	−24	+28	−4	+1	+66	+48	+24	+20	−50	−50	−60	−25	+6	+60	+40	−8	−24	+4	0	−1
(8.5)	+10	−5	0	+13	+22	−18	−36	−12	−10	−10	+30	+15	+5	+10	−20	−5	+5	0	0	−13
(7.6)	+6	0	0	−8	−36	+6	+12	+4	+4	+25	−9	−5	+3	−12	+6	0	0	0	0	+8
(9.2^2)	+16	−14	+2	−7	−5	−33	0	−14	+19	−7	+42	+14	−1	−14	−28	0	−14	−2	0	+7
$(8.3.2)$	−4	+5	0	+4	+12	+24	+12	+16	−8	−5	−27	−10	+1	+8	+10	−1	−2	0	0	−4
$(7.4.2)$	+2	0	0	+7	−13	−8	+10	−6	+11	+2	−12	+6	−3	+4	−2	0	0	0	0	−7
(7.3^2)	−3	0	0	−4	+12	+6	−12	−6	−12	0	+12	0	+3	−3	0	0	0	0	0	+4
$(6.5.2)$	0	0	0	−5	+3	+6	−12	+2	−7	+3	+6	−2	0	0	0	0	0	0	0	+5
$(6.4.3)$	0	0	0	+1	−3	−12	+6	+6	+5	−1	−3	0	0	0	0	0	0	0	0	−1
$(5^2.3)$	0	0	0	+2	−7	+3	+6	−3	0	0	0	0	0	0	0	0	0	0	0	−2
(5.4^2)	0	0	0	−1	+4	+2	−4	0	0	0	0	0	0	0	0	0	0	0	0	+1
(7.2^3)	−2	0	0	0	+5	+5	0	0	−5	0	0	0	+1							
$(6.3.2^2)$	0	0	0	0	−1	−3	0	0	+1											
$(5.4.2^2)$	0	0	0	0	−2	+1														
$(5.3^2.2)$	0	0	0	0	+1															

	$(33\cdot 2^2)$	$(33\cdot 2\cdot 1^2)$	$(32\cdot 23\cdot 1)$	$(3\cdot 25)$	$(33\cdot 14)$	$(32\cdot 2^2\cdot 1^3)$	$(3\cdot 24\cdot 1^3)$	$(32\cdot 2\cdot 1^5)$	$(3\cdot 23\cdot 1^4)$	$(32\cdot 17)$	$(3\cdot 22\cdot 1^6)$	$(3\cdot 2\cdot 1^8)$	$(3\cdot 1^{10})$	$(26\cdot 1)$	$(25\cdot 13)$	$(24\cdot 15)$	$(23\cdot 17)$	$(22\cdot 19)$	$(2\cdot 1^{11})$	(1^{13})
(13)	$+26$	-130	-130	-13	$+65$	$+390$	$+195$	-273	-455	$+52$	$+364$	-117	$+13$	$+13$	-91	$+182$	-156	$+65$	-13	$+1$
$(11\cdot 2)$	-4	$+75$	-2	-9	-43	-104	$+36$	$+97$	$+15$	-19	-45	$+18$	-2	$+9$	-30	$+27$	-9	$+1$		
$(10\cdot 3)$	$+19$	-80	-35	$+7$	$+20$	$+105$	0	-42	-35	$+3$	$+21$	-3	0	-7	$+14$	-7	$+1$			
$(9\cdot 4)$	-20	$+10$	$+40$	-5	$+10$	-30	-15	-6	$+20$	$+2$	-4	0	0	$+5$	-5	$+1$				
$(8\cdot 5)$	$+14$	$+30$	-30	$+3$	-5	-10	$+15$	$+5$	-5	0	0	0	0	-3	$+1$					
$(7\cdot 6)$	-5	-17	$+11$	-1	-2	$+9$	-6	0	0	0	0	0	0	$+1$						
$(9\cdot 2^2)$	$+7$	-21	-7	0	$+7$	$+14$	0	-7	0	$+1$										
$(8\cdot 3\cdot 2)$	-7	$+9$	$+5$	0	-1	-5	0	$+1$												
$(7\cdot 4\cdot 2)$	$+3$	$+6$	-3	0	-2	$+1$														
$(7\cdot 3^2)$	$+2$	-4	0	0	$+1$															
$(6\cdot 5\cdot 2)$	-1	-3	$+1$																	
$(6\cdot 4\cdot 3)$	-2	$+1$																		
$(5^2\cdot 3)$	$+1$																			

On a Certain Envelope. *By* Prof. Wolstenholme, M.A., Sc.D.

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If with a point O of the ellipse $a^2y^2+b^2x^2=a^2b^2$ as centre, we describe a circle such that triangles can be inscribed in the ellipse whose sides touch the circle, the envelope of these circles consists of two distinct curves, one being an ellipse, co-asymptotic with the given ellipse, but increased in the linear ratio $a^2+b^2 : a^2-b^2$ $(a>b)$; and the other a curve of the degree 6 and class 6, which osculates the former in four points.

Taking $(a\cos\theta, b\sin\theta)$ for the point O, and forming the discriminant of

$$k\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+(x-a\cos\theta)^2+(y-b\sin\theta)^2-r^2,$$

we find that $\quad \Delta=\dfrac{1}{a^2b^2}, \quad \Theta=\dfrac{\cos^2\theta}{a^2}+\dfrac{\sin^2\theta}{b^2}+\dfrac{r^2}{a^2b^2},$

$$\Theta'=r^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right), \quad \Delta'=r^2;$$

and, since $\Theta'^2=4\Theta\Delta'$, we get

$$r^2=\frac{4a^2b^2}{(a^2-b^2)^2}(a^2\sin^2\theta+b^2\cos^2\theta), \text{ or } 0.$$

The former value alone is here considered. The equation of the circle is then $(x-a\cos\theta)^2+(y-b\sin\theta)^2=\dfrac{4a^2b^2}{(a^2-b^2)^2}(a^2\sin^2\theta+b^2\cos^2\theta)$; whence for the envelope

$$(x-a\cos\theta)\,a\sin\theta-(y-b\sin\theta)\,b\cos\theta=\frac{4a^2b^2}{a^2-b^2}\sin\theta\cos\theta,$$

or $$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2+\frac{4a^2b^2}{a^2-b^2}=\frac{(a^2+b^2)^2}{a^2-b^2},$$

which is the equation of a normal, at the point whose excentric angle is θ, to the ellipse $\quad \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\left(\dfrac{a^2+b^2}{a^2-b^2}\right)^4.$

The equation of this normal, on which the two points of the envelope lie, may be written $\quad \dfrac{a\,(x-k\,a\cos\theta)}{\cos\theta}=\dfrac{b\,(y+k\,b\sin\theta)}{\sin\theta},$

where $k=\dfrac{a^2+b^2}{a^2-b^2}$, and, if we suppose each member of this equation

$= \lambda$, we have

$$x = \left(\frac{\lambda}{a^2} + k\right) a\cos\theta, \quad y = \left(\frac{\lambda}{b^2} - k\right) b\sin\theta\,;$$

and, substituting these values in the equation of the circle,

$$a^2\cos^2\theta\left(\frac{\lambda}{a^2} + k - 1\right)^2 + b^2\sin^2\theta\left(\frac{\lambda}{b^2} - k - 1\right)^2 = \frac{4a^2b^2}{(a^2-b^2)^2}(a^2\sin^2\theta + b^2\cos^2\theta),$$

a quadratic in λ, whose two roots determine the two points of the envelope. But $k-1 = \frac{2b^2}{a^2-b^2}$, and $k+1 = \frac{2a^2}{a^2-b^2}$, whence it is obvious that one value of λ is 0, and therefore the other is

$$(2-2k\cos 2\theta) \Big/ \left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right), \quad \text{or} \quad \frac{4a^2b^2}{a^2-b^2}\,\frac{a^2\sin^2\theta - b^2\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}.$$

Hence one point of the envelope (which denote by P), corresponding to $\lambda = 0$, is $x = ka\cos\theta$, $y = -kb\sin\theta$, whose locus is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = k^2 \equiv \left(\frac{a^2+b^2}{a^2-b^2}\right)^2;$$

and the second point (Q) is

$$x = \frac{a\cos\theta}{a^2-b^2}\left(a^2+b^2+4b^2\,\frac{a^2\sin^2\theta - b^2\cos^2\theta}{a^2\sin^2\theta + b^2\cos^2\theta}\right),$$

$$y = \frac{b\sin\theta}{b^2-a^2}\left(b^2+a^2+4a^2\,\frac{b^2\cos^2\theta - a^2\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}\right),$$

whose locus is therefore a sextic.

If ϕ be the angle which the normal at O makes with the axis of x, and p the perpendicular from the centre upon the tangent,

$$\frac{a\sin\theta}{\sin\phi} = \frac{b\cos\theta}{\cos\phi} = \frac{ab}{p}\,;$$

whence

$$(a^2-b^2)\,x = \frac{a^2}{p}\cos\phi\,(a^2+b^2-4b^2\cos 2\phi),$$

$$(b^2-a^2)\,y = \frac{b^2}{p}\sin\phi\,(b^2+a^2+4a^2\cos 2\phi).$$

From these we deduce

$$(a^2-b^2)^2(x^2+y^2) = 16a^4b^4\,\frac{\cos^2 2\phi}{p^2} - \frac{8a^2b^2}{p^2}(a^2+b^2)(a^2\cos^2\phi - b^2\sin^2\phi)\cos 2\phi + (a^2+b^2)^2\left(a^2+b^2-\frac{a^2b^2}{p^2}\right);$$

and $2p^2 = a^2+b^2+(a^2-b^2)\cos 2\phi$, whence

$$(a^2-b^2)^2\,(x^2+y^2)$$

$$= (a^2+b^2)^3 + \frac{a^2b^2}{p^2}\{16a^2b^2\cos^2 2\phi - 8\,(a^2+b^2)(a^2\cos^2\phi - b^2\sin^2\phi)\cos 2\phi - (a^2+b^2)^2\}$$

$$= (a^2+b^2)^3 + \frac{a^2b^2}{p^2}\{(a^2+b^2)^2 + 4\,(a^2+b^2)(a^2-b^2+\overline{a^2+b^2}\cos^2\phi)\cos 2\phi - 16a^2b^2\cos^2 2\phi\}$$

$$= (a^2+b^2)^3 + - \frac{a^2b^2}{p^2}\{(a^2+b^2)^2 + 4\,(a^2+b^2)(a^2-b^2)\cos 2\phi + 4\,(a^2-b^2)^2\cos^2 2\phi\}$$

$$= (a^2+b^2)^3 - \frac{a^2b^2}{p^2}(4p^2-a^2-b^2)^2.$$

Hence x^2+y^2 or CQ^2 can never exceed $\dfrac{(a^2+b^2)^3}{(a^2-b^2)^2}$, and can have this maximum value only when $4p^2 = a^2+b^2$, for the possibility of which $a^2 > 3b^2$. [If b^2 be negative, and a^2+b^2 positive, these values of CQ will always be possible, but will be minima.]

Again,
$$(a^2-b^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

$$= \{a^2\cos^2\phi\,(a^2+b^2-4b^2\cos 2\phi)^2 + b^2\sin^2\phi\,(a^2+b^2+4a^2\cos 2\phi)^2\}/p^2$$

$$= (a^2+b^2)^2 - \frac{8a^2b^2}{p^2}(a^2+b^2)\cos^2 2\phi + \frac{16a^2b^2\cos^2 2\phi}{p^2}(a^2+b^2-p^2)$$

$$= (a^2+b^2)^2 + \frac{8a^2b^2}{p^2}(a^2+b^2)\cos^2 2\phi - 16a^2b^2\cos^2 2\phi$$

$$= (a^2+b^2)^2 - \frac{8a^2b^2}{p^2}(a^2-b^2)\cos^3 2\phi;$$

or
$$(a^2-b^2)^2\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{a^2+b^2}{a^2-b^2}\right)^2\right\} = -\frac{8a^2b^2}{p^2}(a^2-b^2)\cos^3 2\phi;$$

showing that the locus of Q *osculates* the locus of P in the four points for which $\cos 2\phi = 0$; *i.e.*, in the four points of contact of a square circumscribing the locus of P.

The tangents to the envelope at P, Q will of course intersect each other in a point (T) on the tangent at O, and will be equally inclined to this tangent, and, the tangents at O, P being equally inclined in opposite directions to the axis, we see that the angles which the normals at P, Q make with the axis will be respectively ϕ, 3ϕ (or $3\phi-\pi$). The coordinates of T are determined by the equations

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1, \quad \frac{x}{a}\cos\theta - \frac{y}{b}\sin\theta = \frac{a^2+b^2}{a^2-b^2},$$

whence we see that T is the pole of the normal at O with respect to the given conic. [Putting $a^2+b^2 = 0$, this gives a pretty property of

the rectangular hyperbola.] The equation of the tangent to the envelope at Q will then be

$$(a^2-b^2)(x\cos 3\phi+y\sin 3\phi)$$

$$= -\frac{4a^2b^2}{p}\cos^2 2\phi+\frac{a^2+b^2}{p}(a^2\cos\phi\cos 3\phi-b^2\sin\phi\sin 3\phi)$$

$$= -\frac{4a^2b^2}{p}\cos^2 2\phi+\frac{a^2+b^2}{2p}(\overline{a^2-b^2}\cos 2\phi+\overline{a^2+b^2}\cos 4\phi)$$

$$= \frac{(a^2-b^2)^2}{p}\cos^2 2\phi-(a^2+b^2)\ \frac{(a^2\sin^2\phi+b^2\cos^2\phi)}{p}$$

$$= \frac{(2p^2-a^2-b^2)^2}{p}-\frac{(a^2+b^2)(a^2+b^2-p^2)}{p} = 4p^3-3p\,(a^2+b^2),$$

or, expressing the second member in terms of ϕ,

$$= \sqrt{a^2\cos^2\phi+b^2\sin^2\phi}\ (\overline{a^2-3b^2}\cos^2\phi+\overline{b^2-3a^2}\sin^2\phi)\,;$$

which equation proves that the curve is of the sixth class.

From this equation it appears that part of the *orthoptic locus* for the sextic is the circle

$$(a^2-b^2)^2\,(x^2+y^2) = (a^2+b^2)^3,$$

which is also the *orthoptic locus* for the ellipse the locus of P. This circle touches the sextic in four points (real only when a^2-3b^2 and a^2+b^2 are both positive), and the normals at these points are both *bitangents* and *binormals*, a peculiarity I do not remember to have met before in any curve.

The *orthoptic locus* is Dr. C. Taylor's name for the locus of the intersection of tangents at right angles to each other.

This curve, by Plücker's formulæ, will therefore have six nodes, six bitangents, four cusps, and four inflexions. When $a^2 > 3b^2$, the six bitangents are all real (two parallel to each axis and two through the centre); but only two of the nodes (on the major axis), there being two acnodes on the minor axis and two at infinity. When $a^2<3b^2$, four bitangents and four nodes are real, the others impossible. The cusps and inflexions appear to be always impossible when b^2 is positive. When b^2 is negative and a^2+b^2 positive, there are two real inflexions, but the cusps are impossible. When b^2 and a^2+b^2 are negative, but $5a^2+b^2$ positive, the cusps are all real, but all the other singularities impossible.

If r', p' be the central radius vector (CQ) and perpendicular on the tangent at Q, we have proved that

$$(a^2-b^2)^2r'^2 = (a^2+b^2)^3-\frac{a^2b^2}{p^2}(4p^2-a^2-b^2)^2\,;$$

and

$$(a^2-b^2)\,p' = 4p^3-3p\,(a^2+b^2).$$

Hence the radius of curvature at Q

$$= r' \frac{dr'}{dp'} = \frac{1}{a^2-b^2} \left\{ 16a^2b^2p - \frac{a^2b^2(a^2+b^2)^2}{p^3} \right\} \Big| (12p^2-3a^2-3b^2)$$

$$= \frac{a^2b^2}{(a^2-b^2)p^3} \frac{16p^4-(a^2+b^2)^2}{3(4p^2-a^2-b^2)} = \frac{a^2b^2}{3(a^2-b^2)} \frac{4p^2+a^2+b^2}{p^3},$$

and the cusps will occur when $4p^2+a^2+b^2=0$. Taking a^2 to be always positive, we see that, since $a^2 > p^2$ at all points of the hyperbola, $5a^2+b^2$ must be positive and a^2+b^2 negative for the cusps to be real; their coordinates being given by

$$x^2(a^2-b^2)^5 = a^4(a^2+5b^2)^3(a^2+b^2),$$

$$y^2(b^2-a^2)^5 = b^4(b^2+5a^2)^3(b^2+a^2);$$

from which the same results might be inferred. I am not quite clear in my mind as to the inflexions. There are two real inflexions when $a^2+b^2=0$, the locus of Q being then a lemniscate of Bernoulli. In general, the radius of curvature is infinite only when p is 0; and when $p=0$, which happens at the points at ∞ on the hyperbola, r' is also infinite, except in the single case when $a^2+b^2=0$. Thus it would seem that there are two real inflexions when the given conic is an hyperbola, but that these are both at infinity; except for the rectangular hyperbola, for which curve they are at the centre.

When the given conic is a parabola, the locus of Q is a curve of degree 4, and class 4, having a real node and real bitangent, and two impossible cusps. This case I have, however, fully investigated in the answer to a question in the *Educational Times*.

I have found the figure of the locus of Q tolerably easy to draw for the two essentially different forms when the given conic is an ellipse, (according as $a^2 >$ or $< 3b^2$); but cannot make a decent figure for the cases when b^2 and a^2+b^2 are negative. There are two points of a high singularity when $5a^2+b^2=0$,

$$x^2 = a^2 \frac{24^3 \times 4}{6^5},\ y=0;\ \textit{i.e.},\ x = \pm\frac{8a}{3},\ y=0.$$

I have thought the investigation of this envelope might be interesting, as the cases of envelopes breaking up in this way are rare. Of course it would be easy to devise envelopes for which it must happen: *e.g.*, a circle with its centre on a parabola and touching the axis will have for the remainder of its envelope a curve (tricusped quartic) which is the involute starting from the vertex of the first negative pedal of the parabola with respect to the focus; but, in the case considered in this paper, there seems no *à priori* reason why the envelope should sever.

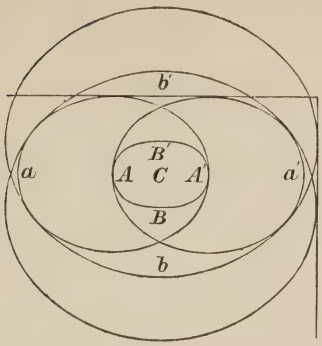


FIG. 1.

(1) a^2-b^2 positive, a^2-3b^2 negative; $a^2 = 2b^2$. The nodes

$$y = 0,\ x = \pm \frac{(a^2+b^2)^{\frac{3}{2}}}{a^2-b^2}\sqrt{\frac{3a^2-b^2}{3a^2+b^2}}, \quad x = 0,\ y = \pm \frac{(a^2+b^2)^{\frac{3}{2}}}{a^2-b^2}\sqrt{\frac{3b^2-a^2}{3b^2+a^2}}$$

are real; and the bitangents

$$x = \pm \frac{a^2}{a^2-b^2}\sqrt{a^2+3b^2}, \quad y = \pm \frac{b^2}{a^2-b^2}\sqrt{b^2+3a^2}$$

are real; the circular points at infinity are the two remaining nodes; and the bitangents through the centre $y = \pm x\frac{b^2}{a^2}\sqrt{\frac{3a^2-b^2}{a^2-3b^2}}$ are impossible. The four cusps and four inflexions are all impossible, AA', BB' the axes of the given ellipse; aa', bb', the axes of the elliptic envelope, osculating the sextic in four points.

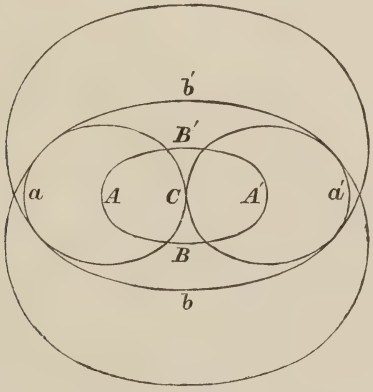


FIG. 2.

(2) $a^2 = 3b^2$. The nodes and bitangents as in (1) are real, but the nodes on the axis of y coincide in the centre; and the bitangents through the centre become real and coincident.

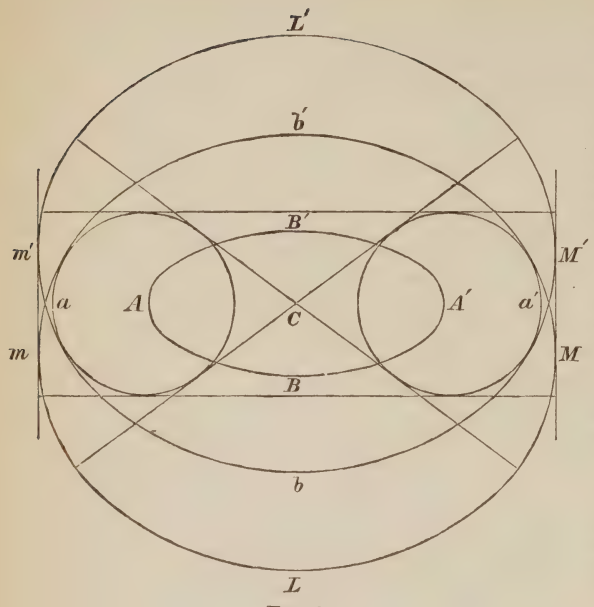


Fig. 3.

(3) $a^2 = 4b^2$. Six bitangents and two nodes are real,

$$AC = CA' = a,\quad BC = CB' = b = \tfrac{1}{2}a;$$
$$aC = Ca' = \tfrac{5}{3}a,\quad bC = Cb' = \tfrac{5}{3}b;$$
$$LC = CL' = \tfrac{11}{3}b.$$

The bitangents through the centre are also binormals. The figure is drawn for $a = 2b$, but these properties hold for all ellipses in which $a^2 > 3b^2$. The distance of the nodes from the centre is

$$\frac{(a^2+b^2)^{\frac{3}{2}}\sqrt{3a^2-b^2}}{(a^2-b^2)\sqrt{3a^2+b^2}};$$

that of the bitangents parallel to the minor axis is

$$\frac{a^2}{a^2-b^2}\sqrt{a^2+3b^2};$$

and that of the bitangents parallel to the major axis is

$$\frac{b^2}{a^2-b^2}\sqrt{3a^2+b^2}.$$

The points of contact M, M', m, m' should be nearer to the transverse axis than is shown in the figure, the distance being in general $\dfrac{b^2\sqrt{3}}{\sqrt{a^2+3b^2}}$.

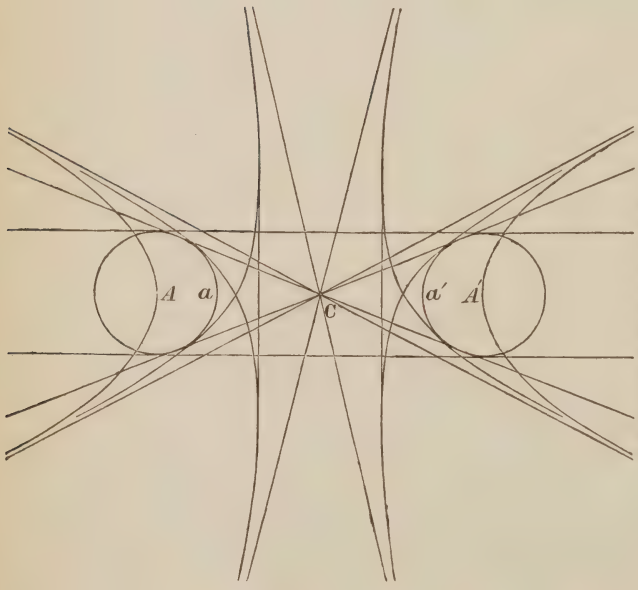


Fig. 4.

(4) b^2 negative, $a^2 + 3b^2$ positive. The locus of Q has six real bitangents and two real nodes; two real asymptotes. The bitangents through the centre are also binormals. $AC = CA' = a$; a, a' the vertices of the locus of P. As $a^2 + 3b^2$ approaches zero, the bitangents parallel to the conjugate axis and the asymptotes tend to coincidence in the conjugate axis. The figure is drawn for $a^2 + 4b^2 = 0$.

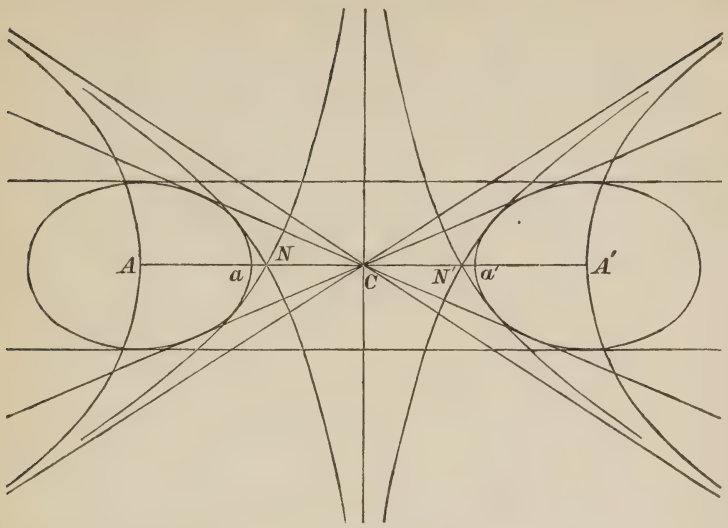
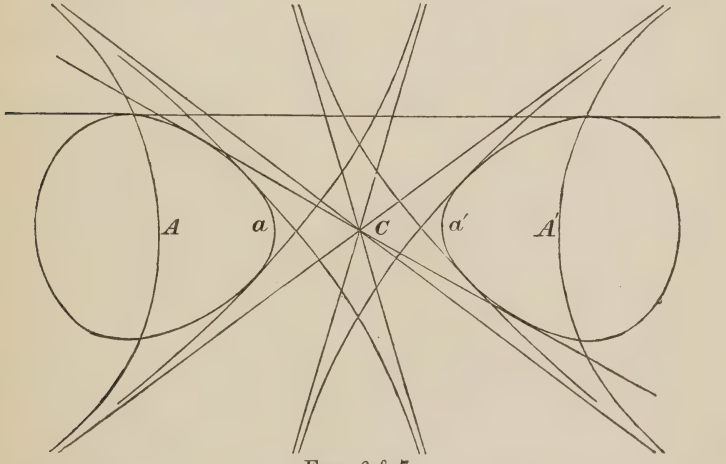


FIG. 5.

(5) $a^2 + 3b^2 = 0$. The bitangents parallel to the conjugate axis coalesce, their points of contact going to infinity; the other bitangents are real, and those through the centre are also binormals, $AC = CA' = a$, $aC = Ca' = \frac{1}{2}a$; $NC = CN' = a\sqrt{\frac{5}{24}}$.



FIGS. 6 & 7.

(6) $a^2 + 3b^2$ negative, $a^2 + b^2$ positive. Four nodes real; four bitangents real, the two parallel to the conjugate axis having become impossible, $AC = CA' = a$, $aC = Ca' = \frac{1}{3}a$. The bitangents through the centre real (only one is shown in the figure), and also binormals. The figure is drawn for $a^2 + 2b^2 = 0$.

(7) $a^2 + b^2 = 0$. There is no need of a figure in this case, the locus of P degenerating into the asymptotes, and the locus of Q being a lemniscate of Bernoulli whose foci coincide with the foci of the hyperbola. The infinite branch disappears and the curve degenerates into a quartic. When $a^2 + b^2$ becomes negative, four real cusps appear, and continue so long as $5a^2 + b^2$ is positive; and a complete system of figures would be for $b^2 + 2a^2 = 0$, $b^2 + 3a^2 = 0$, $b^2 + 4a^2 = 0$, $b^2 + 5a^2 = 0$, $b^2 + 6a^2 = 0$.

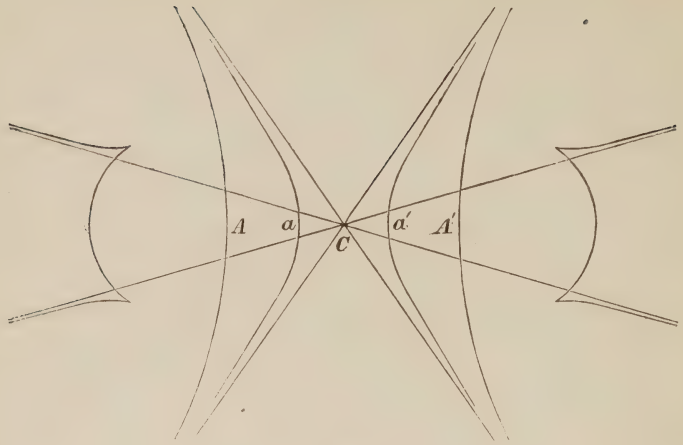


Fig. 8.

(8) a^2+b^2 negative, $3a^2+b^2$ positive; $(2a^2+b^2=0)$. The four cusps all real, the other singularities impossible; the two asymptotes

$$\frac{x}{a\,(a^2+3b^2)} = \pm\frac{y}{b\sqrt{-1}\,(b^2+3a^2)} \text{ real.}$$

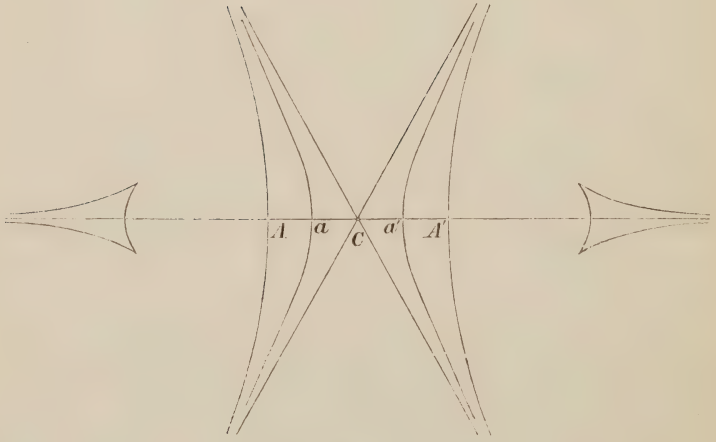


Fig. 9.

(9) $3a^2+b^2=$. Asymptotes become coincident with the axis of x.

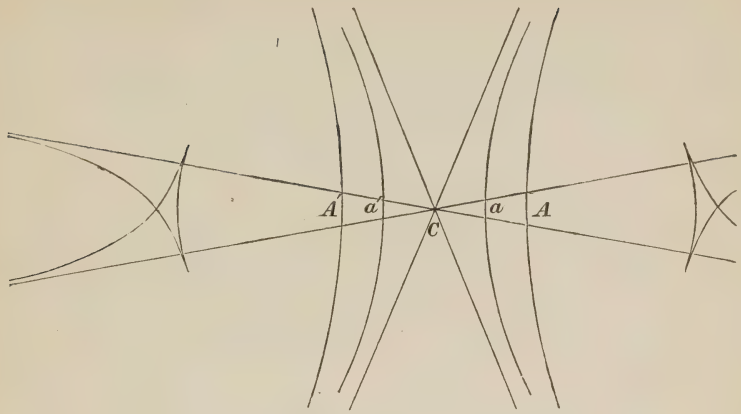


FIG. 10.

(10) $3a^2 + b^2$ negative, $5a^2 + b^2$ positive; $(b^2 + 4a^2 = 0)$. The four cusps real, and the nodes on the axis of x real.

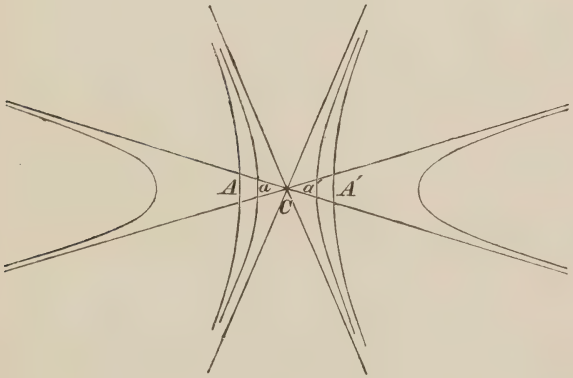


FIG. 11.

(11) $5a^2 + b^2 = 0$. Two cusps and a node coincide at each vertex on the axis of x; each of which points is similar to the point on the curve $y^4 = mx^3$ at the origin. From $b^2 = -5a^2$ to $b^2 = -\infty$, the visible features of the sextic are not much altered.

On Jacobi's Theory of Transformation of Elliptic Functions.

By John Griffiths, M.A.

[*Read Nov. 8th*, 1883.]

It appears to me that Jacobi's theory of transformation may be derived from the equation $\cot\theta\cot\phi = k'$, which is a particular form of the addition-equation for elliptic integrals of the first kind.

The object of the following note is to work out this idea, which leads to what may be called the method of Harmonic Composition.

I start with the two expressions

$$\tan\left(\frac{A-\theta}{2}\right) = \alpha'.\tan\theta \text{ and } \tan\frac{B}{2} = \beta'.\tan\theta,$$

where α', β' are proper fractions. Thus, when $\alpha'=\beta'=$ limiting value unity, we have $A = 3\theta$ and $B = 2\theta$, *i.e.*, pure multiplication. Also, when $\alpha'=\beta'=0$, $A = \theta$ and $B = 0$.

The above expressions can be written in the respective forms

$$\sin A = \frac{\sin\theta\{2\alpha'+1-(1+\alpha')^2\sin^2\theta\}}{1-\alpha^2\sin^2\theta},$$

$$\cos B = \frac{1-(1+\beta'^2)\sin^2\theta}{1-\beta^2\sin^2\theta}; \text{ where } \alpha^2+\alpha'^2 = 1 = \beta^2+\beta'^2.$$

So that, if we put $\sin A = y$ and $\sin\theta = x$, the first is

$$y = \frac{x\{2\alpha'+1-(1+\alpha')^2x^2\}}{1-\alpha^2x^2}.$$

This, in fact, is a form of the cubic transformation-equation.

Similarly, the second expression is

$$y = \frac{1-(1+\beta'^2)x^2}{1-\beta^2x^2},$$

which is a form of the quadric transformation-equation.

From the above, the following results—used in the sequel—are easily deduced, viz.,

$$\cos A = \frac{\sqrt{1-x^2}\{1-(1+\alpha')^2x^2\}}{1-\alpha^2x^2}, \quad \sin B = \frac{2\beta'x\sqrt{1-x^2}}{1-\beta^2x^2},$$

$$\sqrt{1-x^2}\frac{dA}{dx} = 1+\frac{2\alpha'}{1-\alpha^2x^2}, \quad \sqrt{1-x^2}\frac{dB}{dx} = \frac{2\beta'}{1-\beta^2x^2}.$$

It will be seen that the theory which I shall endeavour here to unfold assumes that a higher transformation-equation is made up, so to

speak, of two or more undeveloped lower equations. By an "undeveloped" equation is meant one in which the coefficients are arbitrary, and not fixed to the particular values which would make the expression an actual transformation-equation.

The whole structure is based upon the fact that the different constants may be so determined that the expression

$$M.\sqrt{1-\lambda^2y^2} = \sqrt{1-\kappa^2x^2}\left\{1+\Sigma\frac{2a'}{1-a^2x^2}\right\}$$

can be derived from the integral relation $y = \sin(A+B+C+\ldots)$, which I start with.

As to this, I observe that the equation in question

$$y = \sin(A+B+C+\ldots)$$

gives, when expanded,

$$y = \frac{x(a_0+a_2x^2+a_4x^4+\ldots)}{(1-\alpha^2x^2)(1-\beta^2x^2)\ldots},$$

or, say,

$$y = \frac{Mx\left(1-\frac{\kappa^2}{\alpha^2}x^2\right)\left(1-\frac{\kappa^2}{\beta^2}x^2\right)\ldots}{(1-\alpha^2x^2)(1-\beta^2x^2)\ldots},$$

if certain relations between the arbitrary quantities α, β, γ, &c., are satisfied. This implies that we can simultaneously change x into $\frac{1}{\kappa x}$ and y into $\frac{1}{\lambda y}$, and that, so far, the integral relation may give rise to the differential equation

$$\frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2y^2}} = \frac{Mdx}{\sqrt{1-x^2 \,.\, 1-\kappa^2x^2}}.$$

Again, since $\sqrt{1-y^2} = \cos(A+B+C+\ldots)$, it will be seen without much difficulty that

$$\sqrt{1-y^2} = \sqrt{1-x^2}\ \frac{f(x)}{(1-\alpha^2x^2)(1-\beta^2x^2)\ldots},$$

where $f(x)$ is a rational and integral function. Hence, owing to the possibility of the above simultaneous change of x, y into $\frac{1}{\kappa x}$, $\frac{1}{\lambda y}$, it follows that $\sqrt{1-\lambda^2y^2} = \sqrt{1-\kappa^2x^2}\ \frac{\phi(x)}{(1-\alpha^2x^2)(1-\beta^2x^2)\ldots}$,

where $\phi(x)$ is also a rational and integral function of x; and thus that the expression $M.\sqrt{1-\lambda^2y^2} = \sqrt{1-\kappa^2x^2}\left\{1+\Sigma\frac{2a'}{1-a^2x^2}\right\}$

may be really only another form of

$$y = \sin(A+B+C+\ldots).$$

Of course, after a certain point the details of the present theory be-

come similar to those of Jacobi's, as we should *à priori* expect to find. The case of a transformation-equation corresponding to $y = \sin n\theta$, where n is an odd number, and more especially an odd prime, has been worked out pretty fully by Jacobi and Cayley, but, so far as I know, the analogues of $\sin 2n\theta$ have not been hitherto developed.

I find that with a slight modification of the A-formula the present method can be applied here also.

PART I.—*Analogues of sin nθ when n is an odd integer, particularly an odd prime.*

1. If we combine the foregoing expressions harmonically, we have the quintic transformation-equation in the form

$$y = \sin(A+B) \quad \text{or} \quad y = \frac{x(a_0 - a_2x^2 + a_4x^4)}{(1-\alpha^2x^2)(1-\beta^2x^2)};$$

where
$$a_0 = 1+2(\alpha'+\beta'),$$
$$a_2 = 2\beta'\{1+(1+\alpha')^2\}+\{\beta'^2+1\,.\,2\alpha'+1\}+(1+\alpha')^2$$
$$= 2(1+\alpha'+\beta')\{1+\alpha'\,.\,1+\beta'\}-(\alpha'+\beta')^2,$$
$$a_4 = (1+\alpha'\,.\,1+\beta')^2.$$

Similarly, the septic equation is

$$y = \sin(A+B+C),$$

where
$$\cos C = \frac{1-(1+\gamma'^2)x^2}{1-\gamma^2x^2}, \quad \gamma^2+\gamma'^2 = 1.$$

Generally, for an odd order n, the equation is

$$y = \sin(A+B+C+\ldots).$$

2. If the expression $y = \sin(A+B+C+\ldots)$ be differentiated, we have

$$dy = \cos(A+B+\ldots)\,\Sigma\frac{dA}{dx},$$

or
$$\frac{dy}{\sqrt{1-y^2\,.\,1-\lambda^2y^2}} = M\frac{dx}{\sqrt{1-x^2\,.\,1-\kappa^2x^2}},$$

if
$$M\sqrt{1-\lambda^2y^2} = \sqrt{1-\kappa^2x^2}\;\Sigma\sqrt{1-x^2}\,\frac{dA}{dx}$$
$$= \sqrt{1-\kappa^2x^2}\left\{1+\Sigma\frac{2\alpha'}{1-\alpha^2x^2}\right\},$$

since
$$\sqrt{1-x^2}\,\frac{dA}{dx} = 1+\frac{2\alpha'}{1-\alpha^2x^2}, \quad \sqrt{1-x^2}\,\frac{dB}{dx} = \frac{2\beta'}{1-\beta^2x^2}, \text{ \&c.}$$

Hence, adopting elliptic-function notation, the above gives

$$M\,.\,\mathrm{dn}(Mu,\lambda) = \mathrm{dn}(u,\kappa)\left\{1+\Sigma\frac{2\alpha'}{1-\alpha^2\,\mathrm{sn}^2(u,\kappa)}\right\}.$$

(N.B.—To obtain Professor Cayley's standard forms, write $\frac{1}{M}$ for M throughout the present note.)

The constants α, β, γ, &c. will presently be shown to be identical with Jacobi's coefficients arising from the n-division of the complete function K.

The multiplier M varies in value from n to 1. For example, if α, β, γ, &c. are all zeros, we have pure multiplication or purely trigonometrical formulæ, and $M=n$. On the other hand, if

$$\alpha=\beta=\gamma=\ldots\ldots=1,$$

then $M=1$, since the formula for M in terms of α, β, γ, &c., is $M=1+2\Sigma\alpha'$.

3. As an illustration of the above method, I work out the quintic transformation in detail.

In this case $y=\sin(A+B)$

$$=\frac{x\{2(\alpha'+\beta')+1-a_2x^2+a_4x^4\}}{(1-\alpha^2x^2)(1-\beta^2x^2)};$$

where $a_2=2(1+\alpha'+\beta')(1+\alpha'.1+\beta')-(\alpha'+\beta')^2$,

$a_4=(1+\alpha'.1+\beta')^2$,

or
$$y=\frac{Mx\left(1-\frac{\kappa^2}{\alpha^2}x^2\right)\left(1-\frac{\kappa^2}{\beta^2}x^2\right)}{(1-\alpha^2x^2)(1-\beta^2x^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

if
$$M=1+2(\alpha'+\beta') \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$M\kappa^2\left(\frac{1}{\alpha}+\frac{1}{\beta^2}\right)=2(1+\alpha'+\beta')(1+\alpha'.1+\beta')-(\alpha'+\beta')^2 \quad \ldots(3),$$

$$M\frac{\kappa^4}{\alpha^2\beta^2}=(1+\alpha'.1+\beta')^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Again, if in (1) we change x, y into $\frac{1}{\kappa x}$, $\frac{1}{\lambda y}$, then

$$\frac{1}{\lambda}=\frac{M^2\kappa^3}{(\alpha\beta)^4} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Taking the equations (2), (3), (4), and (5), there are clearly only three unknown quantities, viz., M and α, β, so that there must be a relation between the moduli λ and κ.

If, following Jacobi, we write u^4, v^4 for κ, λ, then, from the above,

$$M=1+2(\alpha'+\beta'),$$

$$\frac{u^2}{v^2}(\alpha^2+\beta^2)=2(1+\alpha'+\beta')(1+\alpha'.1+\beta')-(\alpha'+\beta')^2,$$

$$(1+\alpha')(1+\beta')=\frac{u^5}{v},$$

$$(1-\alpha')(1-\beta')=Muv^3.$$

Hence
$$2\,(1+\alpha'\beta') = \frac{u^5}{v} + Muv^3,$$
$$2\,(\alpha'+\beta') = \frac{u^5}{v} - Muv^3,$$
or
$$M-1 = \frac{u^5}{v} - Muv^3, \quad i.e.,\ M = \frac{\frac{u^5}{v}+1}{1+uv^3}.$$

Again, $\frac{u^2}{v^2}(2-\alpha'^2-\beta'^2) = (M+1)\frac{u^5}{v} - \alpha'^2 - \beta'^2 - \frac{u^5}{v} - Muv^3 + 2$;

since
$$\alpha^2+\alpha'^2 = \beta^2+\beta'^2 = 1;$$

or
$$\left(\frac{u^2}{v^2}-1\right)(2-\alpha'^2-\beta'^2) = M\left(\frac{u^5}{v} - uv^3\right),$$

i.e.,
$$2-\alpha'^2-\beta'^2 = Muv\,(u^2+v^2),$$
$$2\,(1+\alpha'\beta') - (\alpha'+\beta')^2 = Muv\,(u^2+v^2),$$
$$\frac{u^5}{v} + Muv^3 - \left(\frac{M-1}{2}\right)^2 = Muv\,(u^2+v^2),$$
$$\frac{u^5}{v} - Mu^3v = \left(\frac{M-1}{2}\right)^2,$$
$$\frac{u^5}{v} - \frac{\left(\frac{u^5}{v}+1\right)u^3v}{1+uv^3} = \frac{1}{4}\,\frac{u^2}{v^2}\left(\frac{u^4-v^4}{1+uv^3}\right)^2,$$
$$\frac{u^3\,(u^2-v^2)(1-u^3v)}{v\,(1+uv^3)} = \frac{1}{4}\,\frac{u^2}{v^2}\left(\frac{u^4-v^4}{1+uv^3}\right)^2,$$
$$4uv\,(1-u^3v)(1+uv^3) = (u^2-v^2)(u^2+v^2)^2,$$
$$u^6-v^6+5u^2v^2\,(u^2-v^2) - 4uv\,(1-u^4v^4) = 0;$$

the modular equation.

4. The next step is the determination of the coefficients α, β, γ, &c. For this purpose it is essential to find the relation between the complete functions Λ and K.

The integral equation is
$$y = Mx\left\{\frac{1-\frac{\kappa^2}{a^2}x^2}{1-a^2x^2}\right\},$$
where $\{\ \}$ denotes the product arising from the different constants α, β, γ, &c.

If we write $\sin\phi$, $\sin\theta$ for y and x, this leads to the differential equation
$$\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}} = M\,\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}}.$$

Now, when $x=1$ or $\theta=\frac{\pi}{2}$, we have $A=3\frac{\pi}{2}$, $B=C=\ldots=2\frac{\pi}{2}$; hence $\phi=n\frac{\pi}{2}$, and consequently

$$\int_0^{n\frac{1}{2}\pi}\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}}=M\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{1-\kappa^2\sin^2\theta}}$$

(n odd integer), *i.e.*, since

$$\int_0^{n\frac{1}{2}\pi}\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}}=n\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}},$$

the relation required between Λ and K is $n\Lambda=M\mathrm{K}$.

Again, $y=\sin\phi=\mathrm{sn}\,(Mu,\lambda)$ vanishes when $Mu=2s\Lambda$, where s is an integer, *i.e.*, when $u=\frac{2s\mathrm{K}}{n}$. Also the right-hand side of the integral equation vanishes when $\mathrm{sn}\,u=x=\frac{\alpha}{\kappa}$, $x=\frac{\beta}{\kappa}$, &c. Hence, clearly, $\frac{\alpha}{\kappa}$, &c. must be of the form $\mathrm{sn}\frac{2s\mathrm{K}}{n}$; or

$$\alpha=\kappa\,\mathrm{sn}\frac{2s\mathrm{K}}{n}\text{ and }\alpha'=\mathrm{dn}\frac{2s\mathrm{K}}{n}.$$

Moreover, it is seen without difficulty that s may be taken from 1 to $\frac{1}{2}(n-1)$.

It thus appears that Jacobi's first transformation arises from a slight modification of ordinary trigonometrical formulæ, and that the problem of finding the coefficients α, β, γ, &c. is identical with that of constructing two confocal ellipses so related that it is possible to have a polygon of n sides inscribed in the inner and circumscribing the outer curve. (See a note by the present writer on the derivation of elliptic function formulæ from confocal conics—*Proc. Lond. Math. Soc.*, Vol. xiv., p. 46).

PART II.—*Analogues of sin nθ when n is an even integer.*

The foregoing case is that of an odd number n, but the idea started with is applicable when n is an even integer.

For this I take the B-, C-, &c. formulæ as before, and replace the A-formula by Landen's equation, viz., I now write

$$\sin A=\frac{(1+\kappa')\,x\sqrt{1-x^2}}{\sqrt{1-\kappa^2x^2}},\quad\text{and}\quad\cos A=\frac{1-(1+\kappa')\,x^2}{\sqrt{1-\kappa^2x^2}}.$$

The transformation-equation then is, in general,

$$y=\sin\,(A+B+C+\ldots).$$

This gives, in a precisely similar way to the above, the differential equation $$\frac{dy}{\sqrt{1-y^2 \cdot 1-\lambda^2 y^2}} = M \frac{dx}{\sqrt{1-x^2 \cdot 1-\kappa^2 x^2}},$$

if $$M\sqrt{1-\lambda^2 y^2} = \sqrt{1-\kappa^2 x^2}\left\{1+\frac{\kappa'}{1-\kappa^2 x^2}+2\Sigma\frac{\beta'}{1-\beta^2 x^2}\right\},$$

since $\sqrt{1-x^2}\,\frac{dA}{dx}$ is now equal to $1+\frac{\kappa'}{1-\kappa^2 x^2}$, and $\sqrt{1-x^2}\,\frac{dB}{dx}$, &c. have the same values as before.

The following are the results with regard to the equations corresponding to $2n = 2, 4, 6$, respectively :—

1. $2n = 2$, $y = \sin A$: Landen's equation, which is well known. Modular equation $\lambda = \frac{1-\kappa'}{1+\kappa'}$.

2. $2n = 4$: $y = \sin(A+B)$,

i.e., $$y = \frac{x\sqrt{1-x^2}}{\sqrt{1-\kappa^2 x^2}}\,\frac{1+\kappa'+2\beta'-(1+\kappa')(1+\beta')^2 x^2}{(1-\beta^2 x^2)}$$

$$= \frac{Mx\sqrt{1-x^2}}{\sqrt{1-\kappa^2 x^2}}\left(\frac{1-\frac{\kappa^2}{\beta^2}x^2}{1-\beta^2 x^2}\right);$$

if $$M = 1+\kappa'+2\beta',\ M\frac{\kappa^2}{\beta^2} = (1+\kappa')(1+\beta')^2,\ \frac{1}{\lambda} = \frac{M^2}{\beta^4};$$

the last relation being obtained from the change of x, y into $\frac{1}{\kappa x}$ and $\frac{1}{\lambda y}$, respectively.

Hence $$M = (1+\sqrt{\kappa'})^2,$$

$$\beta = \sqrt{1-\kappa'},\quad \sqrt{\lambda} = \frac{1-\sqrt{\kappa'}}{1+\sqrt{\kappa'}}.$$

3. $2n = 6$: here $y = \sin(A+B+C)$, or

$$y = \frac{x\sqrt{1-x^2}}{\sqrt{1-\kappa^2 x^2}}\,\frac{(a_0-a_2 x^2+a_4 x^4)}{(1-\beta^2 x^2)(1-\gamma^2 x^2)},$$

where $$a_0 = 1+\kappa'+2(\beta'+\gamma'),$$

$$a_2 = (1+\kappa')(2+p^2+2p+2q)+2p(1+q),$$

$$a_4 = (1+\kappa')(1+\beta'\cdot 1+\gamma')^2,$$

if p, q be written for $\beta'+\gamma'$ and $\beta'\gamma'$.

The final results are by no means easy to arrive at. I write them down so far as I have been able to derive them.

Modular equation.—If we put

$$\sqrt[4]{\lambda} = v \quad \text{and} \quad \sqrt[4]{\frac{1-k'}{1+k'}} = c',$$

this presents itself in the form

$$v^4 - 2v^3c'^3 + 2vc' - c'^4 = 0,$$

or in the equivalent one

$$\{(1+\lambda)(1+k') - 2\}^4 = 64\lambda k^2\lambda'^2 k'.$$

The Multiplier M.—

$$M = \frac{2}{1+c'^4}\left\{2\frac{c'^3}{v} - 1\right\}, \quad \frac{6}{M} = \frac{2}{1+c^4}\left\{\frac{2c^3}{u'} - 1\right\};$$

where $$\sqrt[4]{\frac{1-v^4}{1+v^4}} = c \quad \text{and} \quad \sqrt[4]{k'} = u'.$$

The most striking peculiarity of the modular equation for an even transformation $2n$ seems to be the interchangeability of λ and k', which enables us to illustrate the theory that a transformation is an incomplete multiplication.

As an instance of this, I take the quartic transformation-equation

$$y = \sin(A+B) = \frac{Mx\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}\left\{\frac{1-(1+k')\,x^2}{1-(1-k')\,x^2}\right\},$$

where $$M = (1+\sqrt{k'})^2, \quad \frac{4}{M} = (1+\sqrt{\lambda})^2;$$

modular equation $(1+\sqrt{\lambda})(1+\sqrt{k'}) = 2$, and differential equation

$$\frac{dy}{\sqrt{1-y^2}\,.\,1-\lambda^2y^2} = M\frac{dx}{\sqrt{1-x^2}\,.\,1-k^2x^2}.$$

Now, if we write

$$z = (1+\sqrt{\lambda})^2\,\frac{y\sqrt{1-y^2}}{\sqrt{1-\lambda^2y^2}}\left\{\frac{1-(1+\lambda)\,y^2}{1-(1-\lambda)\,y^2}\right\},$$

this is transformed, by writing $i\dfrac{z}{\sqrt{1-z^2}}$, $i\dfrac{y}{\sqrt{1-y^2}}$ for z, y, into

$$z = (1+\sqrt{\lambda})^2\,\frac{y\,(1+\lambda y^2)}{1+2\sqrt{\lambda}\,(1+\sqrt{\lambda}+\lambda)\,y^2+\lambda^2y^4},$$

which gives rise to the differential equation

$$\frac{dz}{\sqrt{1-z^2}\,.\,1-k^2z^2} = (1+\sqrt{\lambda})^2\,\frac{dy}{\sqrt{1-y^2}\,.\,1-\lambda^2y^2}.$$

Hence $$\frac{dz}{\sqrt{1-z^2 \,.\, 1-k^2z^2}} = (1+\sqrt{\lambda})^2 \frac{dy}{\sqrt{1-y^2 \,.\, 1-\lambda^2y^2}}$$

$$= (1+\sqrt{\lambda} \,.\, 1+\sqrt{k'})^2 \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}},$$

or $$\frac{dz}{\sqrt{1-z^2 \,.\, 1-k^2z^2}} = 4 \frac{dx}{\sqrt{1-x^2 \,.\, 1-k^2x^2}}.$$

Appendix.

(*a*) As the cubic transformation-equation is a very interesting one, I may mention that it can be deduced, amongst other ways, (1) from Landen's equation, (2) by drawing a normal from a point on an ellipse to intersect a certain circle. I append the latter method.

If we take a point $(a \sin\theta, b \cos\theta)$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the equation of the normal thereat is

$$\frac{ax}{\sin\theta} - \frac{by}{\cos\theta} = a^2-b^2,$$

and, if we seek the intersections of this line with the circle

$$x^2+y^2 = (a-b)^2,$$

we have, writing $x = (a-b)\sin\phi$, $y = (a-b)\cos\phi$,

$$\frac{a\sin\phi}{\sin\theta} - \frac{b\cos\phi}{\cos\theta} = a+b.$$

Now, supposing θ to be given, it is easily found that the roots of ϕ are

$$\pi-\theta, \text{ and } \sin\phi = \frac{\{2a'+1-(1+a')^2\sin^2\theta\}\sin\theta}{1-a^2\sin^2\theta},$$

where $$\frac{b}{a} = a' \quad \text{and} \quad a^2+a'^2 = 1.$$

Hence, in the cubic equation, y or $\sin\phi$ may be found by drawing a normal to an ellipse.

(*b*) The determination of the modular equation in the case of a transformation of an odd order is also a very interesting exercise. It may be noticed that the equation $\frac{u^n}{v} = [1+a']$, which can be deduced without difficulty from the above theory, will enable us to write down four terms of the modular equation. I illustrate this by taking the case of $n = 11$. In the limiting case, *i.e.*, when we have pure multi-

plication, $\alpha'=\beta'=\gamma'=\delta'=\epsilon'=1$ (for $n=11$ there are the five coefficients α', β', &c.), so that, taking u and v to be very small,

$$\frac{u^{11}}{v}=32,$$

and we can start the modular equation thus, viz.,

$$32uv-u^{12}+v^{12}-32u^{11}v^{11}+\ldots=0.$$

This is equivalent to saying that there is skew symmetry, and that, supposing the above to represent a curve, it must be one with a node at the origin ($u=0$, $v=0$), the tangents of which are u and v.

Again, making some further assumptions,—viz., that the equation is unchanged when we write (1) u^{-1}, v^{-1} for u, v; (2) v, $-u$ for u, v; (3) $-k$, $-\lambda$ for k, λ, *i.e.*, $-u^4$, $-v^4$ for u^4, v^4, or say $\frac{i+1}{\sqrt{2}}u$, $\frac{i-1}{\sqrt{2}}v$ for u, v,—we may say that the following is a *possible* modular equation, viz.,

$$\begin{aligned}v^{12}&-(32u^{11}+b_{11}u^3)\,v^{11}+a_{10}u^6v^{10}+(a_9u^9+b_{11}u)\,v^9+a_8u^4v^8\\&+a_7u^7v^7-(a_{10}u^{10}-a_{10}u^2)\,v^6-a_7u^5v^5-a_8u^8v^6-(a_9u^3+b_{11}u^{11})\,v^3\\&-a_{10}u^6v^2+(32u+b_{11}u^9)\,v-u^{12}=0.\end{aligned}$$

Hence, finally, assuming that $(v+1)(v-1)^{11}=0$, when $u=1$, we have

$$b_{11}=-22,\quad a_{10}=44,\quad a_9=-88,\quad a_8=165,\quad a_7=-132.$$

For a transformation of an odd order, the modular equation is of the form

$$\phi(u,v)=\frac{u^4}{v^4}\,\phi\left(\frac{1}{u},\frac{1}{v}\right);$$

and for an even integer, *i.e.*, for an analogue of $\sin 2n\theta$, the functional relation is

$$v^4\phi(u,v)=\phi\left(u,\frac{1}{v}\right);$$

where ϕ denotes a rational function, and $u^4=k$, $v^4=\lambda$.

(*c*) It is evident that the theory, as regards Part II., may be completed by means of the division of the complete function into a number of equal parts. Thus, the formula

$$M.\sqrt{1-\lambda^2y^2}=\sqrt{1-k^2x^2}\left\{1+\frac{k'}{1-k^2x^2}+2\Sigma\frac{\beta'}{1-\beta^2x^2}\right\}$$

may be written

$$M.\,\mathrm{dn}\,(Mu,\lambda)=\mathrm{dn}\,(u,k)\left\{1+\frac{k'}{\mathrm{dn}^2(u,k)}+2\Sigma\frac{\mathrm{dn}\frac{sK}{n}}{1-k^2\,\mathrm{sn}^2\frac{sK}{n}\,\mathrm{sn}^2(u,k)}\right\};$$

where the order of the transformation is $2n$, and s is an integer extending from 1 to $n-1$.

The equation just written gives the value of $\mathrm{cn}(Mu, \lambda)$ in terms of functions to modulus k, viz.,

$$M\lambda.\,\mathrm{cn}(Mu,\lambda) = \pm\,\mathrm{dn}(u,k)\left\{1+\frac{(-)^n k'}{\mathrm{dn}^2(u,k)}+2\Sigma\frac{(-)^s\,\mathrm{dn}\frac{sK}{n}}{1-k^2\,\mathrm{sn}^2\frac{sK}{n}\,\mathrm{sn}^2(u,k)}\right\}$$

In this case also the problem of transformation is brought into relation with that of the confocal ellipses mentioned above.

Thursday, December 13th, 1883.

SAMUEL ROBERTS, F.R.S., Vice-President, in the Chair.

Messrs. A. B. Basset, M.A., Trinity College, Cambridge; H. Fortey, M.A., formerly of Caius College, Cambridge; R. T. Glazebrook, M.A., Fellow and Assistant Tutor, Trinity College, Cambridge; G. Heppel, M.A., St. John's College, Cambridge; J. J. Thomson, M.A., Fellow of Trinity College, Cambridge; H. H. Turner, B.A., Scholar of Trinity College, Cambridge; and Professor W. Thomson, B.A., B.Sc., Stellenbosch College, Cape Colony, were elected Members.

Mrs. Spottiswoode's reply to the Society's Vote of Condolence was read. Professor Greenhill reported that he had examined the Treasurer's accounts, and had found them correct; the Treasurer's report was then received, subject to the verification by the Auditor of the Society's Moneys at the Bank of England.

The following papers were read:

"The form of Standing Waves on the Surface of Running Water:" Lord Rayleigh, F.R.S.

"A Method of finding the Plane Sections of a Surface, and some considerations as to its Extension to Space of more than Three Dimensions:" Mr. W. J. C. Sharp.

"On a Deduction from the Elliptic-Integral Formula $y = \sin(A + B + C + \ldots)$:" Mr. J. Griffiths.

The following presents were received:

Cabinet likeness of Professor William Thomson, Cape Colony.

"Educational Times" for December, 1883.

"An Elementary Treatise on the Planetary Theory," by the late C. H. Cheyne, Edited by Rev. A. Freeman; London, 1883, 8vo.

"An Elementary Treatise on Conic Sections," Part I., by H. G. Willis; 8vo, Cambridge, 1883.

"Beiblätter zu den Annalen der Physik und Chemie," B. vii., St. 11.

"Bulletin de la Société Mathématique de France," T. xii., No. 4.

"Bulletin des Sciences Mathématiques et Astronomiques," for September and October, 1883.

"Bulletin Astronomique and Météorologique de l'Observatoire Impérial de Rio de Janeiro," Nos. 8 and 9.

"Acta Mathematica," ii., 3, ii., 4.

"Archiv for Mathematik og Naturvidenskab," 3 B., 4 H. & Tillægshefte; 4 B., 1, 2, 3, 4 H.; 5 B., 1, 2, 3, 4 H.; 6 B., 1, 2, 3, 4 H.; 7 B., 1, 2, 3, 4 H.; 8 B., 1 and 2 H.

"Et Par synthetiske Methoder især til Brug ved Studiet af metriske Egenskaber," af Elling Holst; 8vo, Christiania.

"Om Algebraiske Differentialligninger, der tilstede infinitesimale Transformationer," af Sophus Lie; 8vo pamphlet.

"Untersuchungen über Differentialgleichungen," iii., von Sophus Lie; 8vo pamphlet.

"Bestimmung aller Raumcurven, deren Krümmungsradius, Torsionsradius und Bogenlänge durch eine beliebige Relation verknüpft sind," von Sophus Lie; 8vo pamphlet.

"Classification der Flächen nach der Transformationsgruppe ihrer geodätischen Curven," von Sophus Lie; 4to, Kristiania, 1878.

The Form of Standing Waves on the Surface of Running Water.

By Lord Rayleigh, D.C.L., F.R.S.

[*Read Dec. 13th*, 1883.]

The present investigation had its origin in an attempt to explain more fully some interesting phenomena described by Scott Russel* and Thomson,† and figured by the former. When a small obstacle, such as a fishing line, is moved forward slowly through still water, or (which of course comes to the same thing) is held stationary in moving water, the surface is covered with a beautiful wave-pattern, fixed relatively to the obstacle. On the up-stream side the wave-length is short, and, as Thomson has shown, the force governing the vibrations is principally cohesion. On the down-stream side the waves are longer, and are governed principally by gravity. Both sets of waves move with the same velocity relatively to the water; namely, that required in order that they may maintain a fixed position relatively to the obstacle. The same condition governs the velocity, and therefore the wave-length, of those parts of the wave-pattern where the fronts are oblique to the direction of motion. If the angle between this direction and the normal to the wave-front be called θ, the velocity of propagation of the waves must be equal to $v_0 \cos\theta$, where

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* *B. A. Report* for 1844.

† *Phil. Mag.*, Nov. 1871.

v_0 represents the velocity of the water relatively to the (fixed) obstacle.

Thomson has shown that, whatever the wave-length may be, the velocity of propagation of waves on the surface of water cannot be less than about 23 centims. per second. The water must run somewhat faster than this in order that the wave-pattern may be formed. Even then the angle θ is subject to a limit defined by $v_0 \cos\theta = 23$, and the curved wave-front has a corresponding asymptote.

The immersed portion of the obstacle disturbs the flow of the liquid independently of the deformation of the surface, and renders the problem in its original form one of great difficulty. We may, however, without altering the essence of the matter, suppose that the disturbance is produced by the application to one point of the surface of a slightly abnormal pressure, such as might be produced by electrical attraction, or by the impact of a small jet of air. Indeed, either of these methods—the latter especially—gives very beautiful wave-patterns.

Even with this simplification, the difficulties remain considerable. It would appear to be a necessary first step to solve the problem in two dimensions; that is, to find the standing wave-form produced in running water by the impact of a sheet of wind, which strikes the surface along a straight line. Of this I have succeeded in obtaining the solution, and it accounts satisfactorily for one of the leading features of the phenomenon,—the existence of the waves of small wave-length only on the up-stream side, and of the waves of greater wave-length only on the down-stream side of the place of disturbance. In terms of this solution, that of the original problem is analytically expressible, since we may imagine the pressure localised round a point to be the result of the superposition of an infinite system of linear pressures, whose lines of action pass through the point, and are distributed equally in every direction. But the expression in terms of an integral is not readily interpretable, and it is even doubtful—see (23)—whether it has a definite limit when the viscosity of the liquid is supposed to be infinitely small. In fact, that element of the integral which represents a system of parallel waves, travelling (perpendicularly to their own fronts) with the minimum velocity, has an infinite coefficient, as might perhaps have been expected from the corresponding problem for sound, where all waves travel with the same velocity. The prominence of this part of the system is a marked feature of the observed wave-pattern.

But, without an exact solution, it is possible to determine the form of the curved wave-fronts, considered as the envelope of a system of straight lines, and thus to obtain from theory a pretty good general idea of the phenomenon as a whole. In Fig. 3 this construction is

carried out for the particular case in which the asymptotes include a right angle.

Let us suppose that deep water, originally in motion with uniform velocity c parallel to the horizontal coordinate x, is disturbed slightly in two dimensions. If ϕ and ψ be the potential and stream functions, we may take

$$\left.\begin{aligned}\phi &= cx + \Sigma\, a\, e^{-kz} \sin(kx+\epsilon) \\ \psi &= cz - \Sigma\, a\, e^{-kz} \cos(kx+\epsilon)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(1).$$

In (1) z is measured downwards from the undisturbed surface, the wave-length is $2\pi/k$, and, for each value of k, a and ϵ are arbitrary. For the velocity at any point, we have, from (1),

$$\tfrac{1}{2}U^2 = \tfrac{1}{2}c^2 + c\,\Sigma\, ka\, e^{-kz} \cos(kx+\epsilon) \quad \ldots\ldots\ldots\ldots\ldots(2).$$

In calculating the pressure, we will suppose that the motion of each element is opposed by a retarding force proportional to the velocity,* of which therefore the components parallel to the axes may be denoted by $-hu$, $-hv$, h being positive. This ("Theory of Sound," § 239) is not inconsistent with the existence of a velocity potential, but we must imagine a bodily force to act throughout the fluid sufficient to maintain the velocity c. The only other force acting within the fluid is gravity. Hence, on the supposition that the motion is steady, the equation for the pressure takes the form

$$\frac{p}{\rho} = \text{const.} + gz - h(\phi - cx) - \tfrac{1}{2}U^2$$

$$= \text{const.} + gz - h\,\Sigma\, a\, e^{-kz} \sin(kx+\epsilon) - c\,\Sigma ka\, e^{-kz} \cos(kx+\epsilon) \ldots\ldots(3).$$

The equation of the surface, found from (1) by putting $\psi = 0$, is

$$cz = \Sigma a \cos(kx+\epsilon) \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Thus, for the variable part of the pressure just below the surface, we get

$$\frac{\delta p}{\rho} = \Sigma a\,(gc^{-1} - kc)\cos(kx+\epsilon) - h\,\Sigma a \sin(kx+\epsilon) \ldots\ldots(5).$$

In passing from (5) to the expression for the pressure ϖ which must act externally upon the surface, we must include the effect of the capillary tension T. The curvature of the surface is

$$c^{-1}\Sigma\, ak^2 \cos(kx+\epsilon),$$

and thus

$$\frac{\varpi}{\rho} = \Sigma a\,(g + T'k^2 - kc^2)\cos(kx+\epsilon) - hc\,\Sigma\, a \sin(kx+\epsilon) \ldots\ldots(6),$$

in which T' is written for T/ρ.

* January, 1884. The dissipative forces here introduced are ultimately supposed to vanish, but without them it did not seem easy to interpret the analytical expressions to which we are led.

If we introduce a new angle ϵ', defined by

$$\tan \epsilon' = \frac{hc}{g + T'k^2 - kc^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

(6) may be written

$$\frac{c\varpi}{\rho} = \Sigma\, a \surd\{(g + T'k^2 - kc^2)^2 + h^2c^2\} \cos(kx + \epsilon + \epsilon') \ldots\ldots\ldots (8).$$

In the problem before us, we are to regard ϖ as given, and thence determine the form of the surface. If we suppose

$$\frac{\varpi}{\rho} = \Sigma\, \beta \cos(kx + e) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

where β and e are given for each value of k, then $\epsilon + \epsilon' = e$, and

$$c\beta = a \surd\{(g + T'k^2 - kc^2)^2 + h^2c^2\}.$$

Accordingly, by (4),

$$z = \Sigma\, \frac{\beta \cos(kx + e - \epsilon')}{\surd\{(g + T'k^2 - kc^2)^2 + h^2c^2\}}$$

$$= \Sigma\, \frac{\beta\,(g + T'k^2 - kc^2)\cos(kx + e) + \beta hc \sin(kx + e)}{(g + T'k^2 - kc^2)^2 + h^2c^2} \quad \ldots\ldots (10),$$

gives the equation to the surface corresponding to the applied pressures (9).

If we suppose that h is small, and limit ourselves to the case of a single train of waves, *i.e.*, to a single value of k, we see that the phases of ϖ and z are in general coincident or opposite, according as $(g + T'k^2 - kc^2)$ is positive or negative. The first case arises when the wave-length is either very great or very small, and then the pressure is in excess over the troughs and in defect over the crests of the waves. The actual velocity of the waves relatively to the water (c) is here less than that of free waves of the given wave-length, *i.e.*,

$$\surd\{g / k + T'k\}.$$

But when the actual velocity c is greater than that of free waves of the given wave-length, $(g + T'k^2 - kc^2)$ is negative, and then the excess of pressure is to be found over the crests, and the defect of pressure over the troughs of the waves. In the case of transition, when c coincides with the velocity of the free waves, the term in h must be retained, and it shows that the place of maximum pressure is now at that *shoulder* of the wave where the water in its forward motion is falling.

In general, when the pressure along the surface is arbitrary, we must have recourse to Fourier's theorem. Thus

$$\frac{\varpi}{\rho} = \phi(x) = \frac{1}{\pi} \int_0^\infty dk \int_{-\infty}^{\infty} \phi(v) \cos k(v - x)\, dv \quad \ldots\ldots\ldots (11),$$

which is of the form (9).

We now suppose that the abnormal pressure is confined to a very narrow strip at $x = 0$, so that $\phi(v) = 0$, except when v is very small. In this case (11) may be written

$$\frac{\varpi}{\rho} = \frac{1}{\pi}\int \phi(v)\,dv \int_0^\infty dk \cos kx = \frac{1}{\pi}\Phi\int_0^\infty dk \cos kx \quad \ldots\ldots(12),$$

if we put Φ for $\int \phi(v)\,dv$.

The corresponding value of z, from (10), is

$$z = \frac{1}{\pi}\Phi\int_0^\infty dk\,\frac{(g+T'k^2-kc^2)\cos kx + hc\sin kx}{(g+T'k^2-kc^2)^2+h^2c^2} \quad \ldots\ldots(13);$$

and this gives the form of surface assumed by the running water when subjected to a small excess of pressure acting over a narrow strip at the origin.

Before entering upon the general integration and interpretation of (13), it may be well to point out its application in the case where the water is originally at rest ($c=0$). The formula (13) then reduces to

$$\frac{1}{\pi}\Phi\int_0^\infty dk\,\frac{\cos kx}{g+T'k^2} = \frac{\Phi\, e^{\mp\sqrt{(g/T')}\,x}}{2\sqrt{(gT')}},$$

the upper sign being taken when x is positive, and the lower when x is negative.

This solution of the statical problem may of course be obtained independently from the differential equation

$$-T'\frac{d^2z}{dx^2} + gz = \frac{\varpi}{\rho}.$$

In the subsequent treatment of (13) it will conduce to brevity if we put unity for c and T', symbols which can always be restored when desirable from considerations of dimensions. We have, then, to consider

$$\int_0^\infty dk\,\frac{(g-k+k^2)\cos kx + h\sin kx}{(g-k+k^2)^2+h^2} \quad \ldots\ldots(14),$$

and it will assume different forms according as the roots of

$$g-k+k^2 = 0$$

are real or imaginary. For the present, we will take the former alternative, which is equivalent to supposing that the velocity of the water exceeds the minimum velocity of propagation of free waves. We assume, accordingly, that

$$g-k+k^2 = (k_1-k)(k_2-k),$$

where

$$k_1k_2 = g, \quad k_1+k_2 = 1.$$

The quantities k_1, k_2 are positive, and we will suppose them to be in

ascending order of magnitude. We thus replace (14) by

$$\int_0^\infty dk\ \frac{(k_1-k)(k_2-k)\cos kx + h\sin kx}{(k_1-k)^2(k_2-k)^2+h^2} \quad \ldots\ldots\ldots\ldots (15),$$

and of this integral we shall require only the limiting form when $h=0$, as we do not propose to consider in general the effect of finite dissipative forces. On this understanding, the first part of the integral may be replaced at once by

$$\int_0^\infty dk\ \frac{\cos kx}{(k_1-k)(k_2-k)} = \frac{1}{k_2-k_1}\left\{\int_0^\infty \frac{\cos kx\ dk}{k_1-k} - \int_0^\infty \frac{\cos kx\ dk}{k_2-k}\right\} \ldots (16).$$

The integrals which make up (16) are even functions of x, *i.e.*, they take the same arithmetical values whether x be positive or negative. For distinctness, we will suppose that x is positive. Now

$$\int_0^\infty \frac{\cos kx\,dk}{k_1-k} = \int_{-\infty}^{k_1 x} \frac{\cos(k_1x-u)\,du}{u}$$

$$= \cos k_1x \int_{-\infty}^{k_1x} \frac{\cos u\,du}{u} + \sin k_1x \int_{-\infty}^{k_1x} \frac{\sin u\,du}{u},$$

in which

$$\int_{-\infty}^{k_1x} \frac{\cos u\,du}{u} = -\int_{k_1x}^{\infty} \frac{\cos u\,du}{u} = \mathrm{ci}\,(k_1x),$$

$$\int_{-\infty}^{k_1x} \frac{\sin u\,du}{u} = \tfrac{1}{2}\pi + \int_0^{k_1x} \frac{\sin u\,du}{u} = \tfrac{1}{2}\pi + \mathrm{si}\,(k_1x).$$

The functions ci and si may be regarded as known functions, and have been fully tabulated by Glaisher.* Thus

$$\int_0^\infty \frac{\cos kx\,dk}{k_1-k} = \cos k_1x\ \mathrm{ci}\,(k_1x) + \sin k_1x\ \{\tfrac{1}{2}\pi + \mathrm{si}\,(k_1x)\} \ldots\ldots (17),$$

and

$$\int_0^\infty \frac{\cos kx\,dk}{(k_1-k)(k_2-k)} = \frac{1}{k_2-k_1}\left[\begin{array}{l}\cos k_1x\ \mathrm{ci}\,k_1x + \sin k_1x\ (\tfrac{1}{2}\pi + \mathrm{si}\,k_1x) \\ -\cos k_2x\ \mathrm{ci}\,k_2x - \sin k_2x\ (\tfrac{1}{2}\pi + \mathrm{si}\,k_2x)\end{array}\right]$$

$$\ldots\ldots\ldots\ldots (18).$$

When $k_1x = \infty$, $\quad \mathrm{ci}\,k_1x = 0, \quad \mathrm{si}\,k_1x = \tfrac{1}{2}\pi$;

when $k_1x = 0$, $\quad \mathrm{ci}\,k_1x = -\infty, \quad \mathrm{si}\,x = 0.$

In the latter case the limiting form for ci (k_1x) is

$$\mathrm{ci}\,(k_1x) = \gamma + \tfrac{1}{4}\log_e (k_1x)^4 - \tfrac{1}{2}\frac{k_1^2x^2}{1\cdot 2} + \ldots \quad \ldots\ldots\ldots (19),$$

so that, when $x=0$,

$$(18) = \frac{1}{k_2-k_1}\log\frac{k_1}{k_2},$$

which is finite in all cases.

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* "Tables of the numerical values of the Sine-integral, Cosine-integral, and Exponential-integral." (*Phil. Trans.*, 1870.)

When $k_2 = k_1$, (18) changes its form and is replaced by

$$-\frac{d}{dk}\{\cos kx \operatorname{ci} kx + \sin kx\,(\tfrac{1}{2}\pi + \operatorname{si} kx)\},$$

that is, $$x\left\{\sin kx \operatorname{ci} kx - \cos kx\,(\tfrac{1}{2}\pi + \operatorname{si} kx) - \frac{1}{kx}\right\} \quad \ldots\ldots(20).$$

We have now to consider the second part of (15), that is, the limit when $h = 0$ of $$\int_0^\infty \frac{h \sin kx\, dk}{(k_1-k)^2\,(k_2-k)^2+h^2} \quad \ldots\ldots(21).$$

With respect to this, it is evident that the only elements of the integral which contribute to the limiting value are those for which the denominator vanishes with h, *i.e.*, those lying in the immediate neighbourhood of the roots k_1 and k_2. Thus, as k passes through k_1 we may put (k_2-k) equal to (k_2-k_1), and as k passes through k_2 we may put (k_1-k) equal to (k_1-k_2). Hence the limit of (21) is the same as the limit of

$$\int_0^\infty \frac{h \sin kx\, dk}{(k_1-k)^2\,(k_2-k_1)^2+h^2} + \int_0^\infty \frac{h \sin kx\, dk}{(k_1-k_2)^2\,(k_2-k)^2+h^2},$$

or the same as the limit of

$$\frac{1}{k_2-k_1}\int_0^\infty \frac{h' \sin kx\, dk}{(k_1-k)^2+h'^2} + \frac{1}{k_2-k_1}\int_0^\infty \frac{h'' \sin kx\, dk}{(k_2-k)^2+h''^2},$$

in which, k_2 being greater than k_1, h' and h'' are positive, and are supposed ultimately to vanish. Now

$$\int_0^\infty \frac{h' \sin kx\, dk}{(k_1-k)^2+h'^2} = \sin k_1 x \int_{-\infty}^{k_1} \frac{h' \cos ux\, du}{u^2+h'^2} - \cos k_1 x \int_{-\infty}^{k_1} \frac{h' \sin ux\, du}{u^2+h'^2};$$

and $$\lim. \int_{-\infty}^{k_1} \frac{h' \sin ux\, du}{u^2+h'^2} = \lim. \int_{-\infty}^{+\infty} \frac{h' \sin ux\, du}{u^2+h'^2} = 0,$$

$$\lim. \int_{-\infty}^{k_1} \frac{h' \cos ux\, du}{u^2+h'^2} = \lim. \int_{-\infty}^{+\infty} \frac{h' \cos ux\, du}{u^2+h'^2}$$

$$= \lim \int_{-\infty}^{+\infty} \frac{\cos (h'xv)\, dv}{1+v^2} = \lim.\ \pi e^{-h'x} = \pi.$$

Accordingly, the limit of (21) is

$$\frac{\pi \sin k_1 x + \pi \sin k_2 x}{k_2-k_1} \quad \ldots\ldots(22),$$

and retains the same form whether x be positive or negative.

It is evident that in the case of equal roots (22), unlike (18), becomes infinite, so that the retention of h is necessary for a practical result. It is not difficult to show that, when h is very small,

$$\int_0^\infty \frac{h \sin kx\, dk}{(k_1-k)^4+h^2} = \frac{\pi \sin k_1 x}{\sqrt{(2h)}} \quad \ldots\ldots(23),$$

which therefore represents for this case the leading term of the complete expression (15).

Combining (18) and (22), we see that, when x is large and positive, the value of (15) is

$$\frac{2\pi \sin k_1 x}{k_2 - k_1} \quad \ldots\ldots\ldots\

value of $$\cos k_1 x \operatorname{ci} k_1 x + \sin k_1 x \,(\operatorname{si} k_1 x - \tfrac{1}{2}\pi)$$
$$-\cos 2k_1 x \operatorname{ci} 2k_1 x - \sin 2k_1 x \,(\operatorname{si} 2k_1 x - \tfrac{1}{2}\pi) \quad \ldots\ldots\ldots(28)$$

is shown by curve B (Fig. 1), and the ordinates are to have the same value when x is negative as when x is positive. The part near the origin is filled in from the approximate analytical value

$$-\log_e 2 + \tfrac{1}{2}\pi k_1 x.$$

The wave-form is now easily deduced, and is shown in Fig. 2. On the positive side we are to add to (28) $2\pi \sin k_1 x$, and on the negative side we are to add $2\pi \sin 2k_1 x$. It will be seen from the analytical expression that the inclination to the horizon (as well as the depression itself) is continuous in passing through the origin. In fact, this is about the only visible effect of the terms in (28), whose influence, except in the immediate neighbourhood of the origin, is exceedingly small.

We now pass to the consideration of the effect of a pressure localized near a point, instead of distributed along a line. The wave-form is to be found by the superposition of an infinite series of systems similar to (24), (25), at various degrees of obliquity (θ), and of such wave-lengths that $$v_0 \cos\theta = v,$$

v being the velocity perpendicular to the wave-front in each case, and v_0 the velocity of water (previously denoted by c). Now

$$v^2 = kT' + gk^{-1};$$

thus the relation between k and θ is

$$v_0^2 \cos^2\theta = kT' + gk^{-1} \quad \ldots\ldots\ldots(29).$$

By (23) and (24), we see that the crests of the component trains are situated at distances from the origin equal to $(m + \frac{1}{4})\lambda$, where m is an integer. The various wave-fronts thus form a system of similar and similarly situated curves, whose shape is defined as the envelope of a system of straight lines, the perpendicular on which from the origin is equal to p, and is inclined at an angle θ to the direction of the stream, the relation between p and θ being

$$p^2 + \frac{v_0^2 \cos^2\theta}{g}\, p + \frac{T'}{g} = 0 \quad \ldots\ldots\ldots(30).$$

In the case of water, we have in C. G. S. measure $T'/g = \cdot 073$, so that

$$p^2 + \frac{v_0^2 \cos^2\theta}{g}\, p + \cdot 073 = 0 \quad \ldots\ldots\ldots(31).$$

The roots are equal when

$$v_0^2 \cos^2\theta / g = 2\sqrt{(\cdot 073)} \quad \ldots\ldots\ldots(32).$$

The case proposed for consideration is that in which the asymptotes include an angle of 90°, so that the maximum value of θ is 45°. Substituting this in (32), we find, $v_0^2/g = 1{\cdot}081$, and thus

$$p = {\cdot}5405\cos^2\theta \pm \sqrt{\{{\cdot}5405^2\cos^4\theta - {\cdot}0730\}} \quad \ldots\ldots\ldots\ldots (33).$$

From this equation we may calculate any number of corresponding values of p and θ, and thus draw the tangents of which the required curves are the envelopes. The annexed table contains a few such pairs of values, sufficient for an indication of the forms of the curves:—

θ	p_1	p_2	$\cos\theta$	$p_1/\cos\theta$	$p_2/\cos\theta$
0°	1·0086	·0724	1·0000	1·0086	·0724
9°	·9800	·0744	·9877	·9922	·0753
18°	·8964	·0814	·9511	·9425	·0856
27°	·7625	·0957	·8910	·8557	·1074
36°	·5823	·1253	·8090	·7196	·1549
45°	·2702	·2702	·7071	·3821	·3821

The two last columns give the intercepts on the axis, by means of which and the value of $\cos\theta$ the lines are more readily drawn than from the perpendiculars themselves. The result is shown in Fig. 3.

On a Deduction from the Elliptic-Integral Formula $y = \sin(A+B+C+\ldots)$. *By* John Griffiths, M.A.

[*Read Dec. 13th*, 1883.]

The transformation-equation $y = \sin(A+B+C+\ldots)$ for elliptic integrals of the first kind, recently communicated by me to the Society, gives also the relations between the corresponding integrals of the second kind.

1. *Case of a transformation of odd order n.*

Here we have $\sin A = \dfrac{x\{2a'+1-(1+a')^2x^2\}}{1-a^2x^2}$,

$$\cos B = \frac{1-(1+\beta'^2)\,x^2}{1-\beta^2x^2}, \quad \cos C = \frac{1-(1+\gamma'^2)\,x^2}{1-\gamma^2x^2}, \text{ \&c.,}$$

where a, β, γ, &c. are identical with Jacobi's coefficients arising from the n-division of the complete function K; viz.,

$$a \equiv k\,\mathrm{sn}\frac{2sK}{n}, \text{ \&c.}; \qquad a' \equiv \mathrm{dn}\frac{2sK}{n},$$

s being an integer from 1 to $\frac{1}{2}(n-1)$. This being so, the equation $y = \sin(A+B+C+\ldots)$ leads to the relation

$$\frac{d\eta}{\sqrt{1-y^2.\,1-\lambda^2y^2}} = M.\,\frac{dx}{\sqrt{1-x^2.\,1-k^2x^2}},$$

or, say,

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1-\lambda^2\sin^2\phi}} = M\int_0^{\theta} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}\,; \quad i.e.,\ v = Mu,$$

if $$M\,\mathrm{dn}\,(v,\lambda) = \mathrm{dn}\,(u,k)\left\{1+\Sigma\frac{2a'}{1-a^2\,\mathrm{sn}^2\,(u,k)}\right\}.$$

This latter equation is, in fact, another form of the transformation-equation we have started with.

Now, if $\sin A = \dfrac{\mathrm{sn}\,u\,\{2a'+1-(1+a')^2\,\mathrm{sn}^2\,u\}}{1-a^2\,\mathrm{sn}^2\,u}$, where $a^2+a'^2=1$,

then $$\frac{dA}{dk} = \frac{2\,\mathrm{sn}\,u\,\mathrm{cn}\,u}{1-a^2\,\mathrm{sn}^2\,u}\frac{da'}{dk} + \left\{\frac{1}{\mathrm{cn}\,u} + \frac{2a'}{\mathrm{cn}\,u\,(1-a^2\,\mathrm{sn}^2\,u)}\right\}\frac{d}{dk}(\mathrm{sn}\,u);$$

and, similarly,

$$\frac{dB}{dk} = \frac{2\,\mathrm{sn}\,u\,\mathrm{cn}\,u}{1-\beta^2\,\mathrm{sn}^2\,u}\frac{d\beta'}{dk} + \frac{2\beta'}{\mathrm{cn}\,u\,(1-\beta^2\,\mathrm{sn}^2\,u)}\frac{d}{dk}(\mathrm{sn}\,u),$$

&c. &c.

Hence $$\Sigma\frac{dA}{dk} = 2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\Sigma\frac{\dfrac{da'}{dk}}{1-a^2\,\mathrm{sn}^2\,u} + M\frac{\mathrm{dn}\,(v,\lambda)}{\mathrm{cn}\,u\,\mathrm{dn}\,u}\frac{d}{dk}(\mathrm{sn}\,u),$$

and we arrive at the formula

$$\frac{dv}{d\lambda} = \frac{dk}{d\lambda}\left\{M\frac{du}{dk} - \frac{2\,\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,(v,\lambda)}\Sigma\frac{\dfrac{da'}{dk}}{1-a^2\,\mathrm{sn}^2\,u}\right\} + u\frac{dM}{d\lambda},$$

or, since $$M^2 = \frac{nk\,k'^2}{\lambda\,\lambda'^2}\cdot\frac{d\lambda}{dk}, \qquad kk'^2\frac{du}{dk} = E - k'^2F - k^2\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u},$$

we may write

$$E(v,\lambda) - \lambda'^2 v - \lambda^2\frac{\mathrm{sn}\,(v,\lambda)\,\mathrm{cn}\,(v,\lambda)}{\mathrm{dn}\,(v,\lambda)} = \frac{n}{M}\left\{E(u,k) - k'^2u - k^2\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,u}\right\}$$

$$-2\frac{nk\,k'^2}{M^2}\frac{\mathrm{sn}\,u\,\mathrm{cn}\,u}{\mathrm{dn}\,(v,\lambda)}\Sigma\frac{\dfrac{da'}{dk}}{1-a^2\,\mathrm{sn}^2\,u}$$

where $v = Mu$, $$+\lambda\lambda'^2u\frac{dM}{d\lambda};$$

and $$M\,\mathrm{dn}\,(v,\lambda) = \mathrm{dn}\,u\left\{1+\Sigma\frac{2a'}{1-a^2\,\mathrm{sn}^2\,u}\right\}.$$

When $u = K$, $v = n\Lambda$, we have, as a consequence of the above,

$$E(n\Lambda, \lambda) - \lambda'^2 . n\Lambda = \frac{n}{M}\{E(K,k) - k'^2 K\} + \lambda\lambda'^2 K \frac{dM}{d\lambda};$$

i.e.,
$$E(\Lambda, \lambda) - \lambda'^2 \Lambda = \frac{1}{M}\{E(K,k) - k'^2 K\} + \lambda\lambda'^2 \frac{K}{n}\frac{dM}{d\lambda}.$$

If $\frac{1}{M}$ be written for M, the relation in question between the complete functions becomes identical with that given by Professor Cayley. (See "Elliptic Functions," p. 223.)

2. *Case of a transformation of an even order* $2n$.

Here, in the formula $y = \sin(A+B+C+\ldots)$, we have to replace the former expression for A by the following,

viz., $$\sin A = (1+k')\frac{x\sqrt{1-x^2}}{1-k^2x^2},$$

which gives $$\frac{dA}{dk} = -\frac{k}{k'}\frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}^2 u} + \left\{1 + \frac{k'}{\operatorname{dn}^2 u}\right\}\frac{\frac{d}{dk}(\operatorname{sn} u)}{\operatorname{cn} u}.$$

Hence

$$\frac{dv}{d\lambda} = \frac{dk}{d\lambda}\left\{M\frac{du}{dk} - \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn}(v,\lambda)}\left(\frac{\frac{dk'}{dk}}{\operatorname{dn}^2 u} + \Sigma\frac{2\frac{d\beta'}{dk}}{1-\beta^2 \operatorname{sn}^2 u}\right)\right\} + u\frac{dM}{d\lambda};$$

where $$\frac{M \operatorname{dn}(v,\lambda)}{\operatorname{dn} u} = 1 + \frac{k'}{\operatorname{dn}^2 u} + 2\Sigma\frac{\beta'}{1-\beta^2 \operatorname{sn}^2 u},$$

$$M = 1 + k' + 2\Sigma\beta', \quad v = Mu, \quad \beta \equiv k \operatorname{sn}\frac{sK}{n}, \text{ \&c.},$$

s being an integer from 1 to $n-1$; and we can deduce a formula similar to that given above for the corresponding elliptic integrals of the second kind, $E(v,\lambda)$ and $E(u,k)$.

As a geometrical result of these formulæ, it follows that an arc of a given ellipse can, in various ways, be expressed in terms of arcs of two other ellipses.

Note by Professor Cayley.

Consider, for instance, the cubic transformation

$$y = \frac{x[1+2\alpha'-(1+\alpha')^2x^2]}{1-\alpha^2x^2}$$

This implies

$$\sqrt{1-y^2} = \frac{\sqrt{1-x^2}[1-(1+\alpha')^2x^2]}{1-\alpha^2x^2}$$

viz., $\sqrt{1-y^2}$ = a rational multiple of $\sqrt{1-x^2}$;

and the quadric transformation

$$z = \frac{1-(1+\beta'^2)x^2}{1-\beta^2x^2}.$$

This implies

$$\sqrt{1-z^2} = \frac{\sqrt{1-x^2}\,.\,2\beta' x}{1-\beta^2x^2},$$

viz., $\sqrt{1-z^2}$ = a rational multiple of $\sqrt{1-x^2}$.

Hence, assuming

$$u = yz - \sqrt{1-y^2}\sqrt{1-z^2},$$

which is a rational function $= \dfrac{x(a_0+a_2x^2+a_4x^4)}{1-\alpha^2x^2\,.\,1-\beta^2y^2}$,

we have

$$\sqrt{1-u^2} = y\sqrt{1-z^2} + z\sqrt{1-y^2},$$

which is $= \sqrt{1-x^2}$ into a like rational function.

That is, in defining the a_0, a_2, a_4, functions of the two arbitrary coefficients α, β, as above, we have in effect so determined them that $\sqrt{1-u^2}$ shall be $= \sqrt{1-x^2}$ into a rational function of x.

We can then further determine a_0, a_2, a_4 in such wise that the change x into $\dfrac{1}{kx}$ shall change u into $\dfrac{1}{\lambda u}$; and, this being so, making the change in $\sqrt{1-u^2}$, we obtain $\sqrt{1-\lambda^2u^2}$ in the form $\sqrt{1-k^2x^2}$ into a rational function of x; viz., u is a function of x such that

$$\frac{du}{\sqrt{1-u^2\,.\,1-\lambda^2u^2}} = \frac{M\,du}{\sqrt{1-x^2\,.\,1-k^2x^2}}.$$

The theory is thus in effect Jacobi's—with the *novelty* of so combining two lower transformations in such wise that the assumed expression for u as a rational function of x shall give

$$\sqrt{1-u^2} = \sqrt{1-x^2} \text{ into a rational function of } x.$$

It is not necessary that the equations

$$y = \text{rational function of } x \quad \text{and} \quad z = \text{rational function of } x$$

should be elliptic-function transformations. All that is required is that they should be such as to give $\sqrt{1-y^2}$ and $\sqrt{1-z^2}$ each $= \sqrt{1-x^2}$ into rational function of x.

Thursday, January 10th, 1884.

S. ROBERTS, Esq., F.R.S., Vice-President, in the Chair.

Messrs. D. Brockelbank and Asûtosh Mukhopâdhyây, B.A., Calcutta University, were elected Members, and Messrs. Fortey and Heppel were admitted into the Society.

A letter from Prof. Greenhill (the Auditor) was read, in which he stated that he had verified the amount of the moneys standing in the Society's name at the Bank of England.*

The Chairman spoke upon the loss the Society had sustained by the death of Mr. Merrifield, and read the vote of condolence which he, as President, was requested to send to the bereaved family in the name of the Council.

Mr. Buchheim stated "An extension of Pascal's Theorem to Space of Three Dimensions," and communicated a paper "On the Theory of Screws in Elliptic Space"; and Mr. Fortey read a paper "On Contacts and Isolations, a problem in Permutations." Mr. Tucker presented a paper by Prof. H. Lamb, "On the Induction of Electric Currents in Cylindrical and Spherical Conductors," and spoke "On a Group of Circles which are connected with the 'Triplicate-Ratio' Circle."

The following presents were received:—

"Educational Times" for January, 1884.

"Cambridge Philosophical Society—Transactions," Vol. xiii., Pt. iii., 1883.

"Cambridge Philosophical Society—Proceedings," Vol. iv., Pt. vi., 1883.

"Johns Hopkins University Circulars," Vol. iii., No. 27.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial, de Rio de Janeiro," No. 7, July, 1883.

"Atti della R. Accademia dei Lincei—Transunti," Vol. vii, Fasc. 15 and 16.

"Jornal de Sciencias Mathematicas e Astronomicas," publicado pelo Dr. F. Gomes Teixeira, Vol. v., Nos. 1 and 2, 8vo; Coimbra.

"Beiblätter zu den Annalen der Physik und Chemie," B. vii., St. 12.

"Jahrbuch über die Fortschritte der Mathematik, &c." xiii., 2, Jahrgang 1881.

"Bulletin des Sciences Mathématiques et Astronomiques," T. vii., Nov. 1883.

"Anuario del Observatorio Astronómico Nacional de Tacubaya," para el 1884; Mexico, 1883.

"Nieuw Archief voor Wiskunde," Deel i. to Deel x., St. i., 8vo; Amsterdam, 1875 to 1883.

"The Mathematical Magazine: a Journal of Elementary Mathematics," edited by Artemas Martin, M.A., Vol. i., No. 8, Oct. 1883; Erie, Pa.

"American Journal of Mathematics," Vol. vi., No. 2, Dec., 1883; Baltimore.

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* See p. 68.

On the Theory of Screws in Elliptic Space.

By Arthur Buchheim, M.A.

[*Read Jan. 10th*, 1884.]

Introduction.*

[As Grassmann's methods are not as well known as they should be, it may perhaps be convenient to the reader if I prefix a short account of them to my paper; I have not chosen the arrangement which would be best in itself, but that which connects itself most simply with familiar conceptions.

Let the coordinates of a point P in space be $(xyzw)$, referred to any tetrahedron $ABCD$, and let the coordinates be so chosen that

$$x+y+z+w=1;$$

then the fundamental idea of the *Ausdehnungslehre* is that this can be expressed in the form of an equation

$$P=xA+yB+zC+wD \quad (*).$$

If we choose, we can consider this equation as nothing more than a brief mode of expressing the statement immediately preceding it, but in Grassmann's view of the matter the equation (*) is fundamental, and we can, if we choose, express it in words by the statement immediately preceding it.

I now alter the notation so as to make it agree with Grassmann's: we call the vertices of the tetrahedron of reference e_1, e_2, e_3, e_4; the coordinates of a point are written x_1, x_2, x_3, x_4, and the point with these coordinates is called x; we therefore have

$$x=x_1e_1+x_2e_2+x_3e_3+x_4e_4.$$

In the same way, if y be any other point, we have

$$y=y_1e_1+y_2e_2+y_3e_3+y_4e_4.$$

Now an essential part of the *Ausdehnungslehre* is the *multiplication of points*. If we take the two equations just written down, and multiply the right-hand sides and the left-hand sides together in the most obvious way (remembering that multiplication is not to be taken as commutative until we have proved or explicitly assumed it to be so), we get a result which may be written

$$xy=\sum_{i=1}^{i=4}\sum_{k=1}^{k=4}x_iy_ke_ie_k;$$

* This introduction has been drawn up by request of one of the referees.

thus, for example, we have on the right-hand side a term $x_1y_4e_1e_4$, and also a term $x_4y_1e_4e_1$, and any reduction of the expression must depend on the law of multiplication assumed for the four *units* $e_1e_2e_3e_4$. The law assumed by Grassmann is that known as *polar* multiplication; that is, we have

$$e_ie_k = -e_ke_i,$$

$$e_i^2 = 0.$$

Using these equations, we get at once

$$\begin{aligned} xy = \quad & (x_1y_4 - x_4y_1)\, e_1e_4 + (x_2y_4 - x_4y_2)\, e_2e_4 + (x_3y_4 - x_4y_3)\, e_3e_4 \\ & + (x_2y_3 - x_3y_2)\, e_2e_3 + (x_3y_1 - x_1y_3)\, e_3e_1 + (x_1y_2 - x_2y_1)\, e_1e_2. \end{aligned}$$

But, if we consider the coefficients of e_1e_4, &c., we see at once that they are the six coordinates of the line joining xy; and we may therefore say that *the product of two points is the line joining them.*

In precisely the same way, if we take three points x, y, z, and form their product, we find

$$\begin{aligned} xyz = & \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix} e_2e_3e_4 + \begin{vmatrix} x_3 & x_1 & x_4 \\ y_3 & y_1 & y_4 \\ z_3 & z_1 & z_4 \end{vmatrix} e_3e_1e_4 + \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix} e_1e_2e_4 \\ & + \begin{vmatrix} x_3 & x_2 & x_1 \\ y_3 & y_2 & y_1 \\ z_3 & z_2 & z_1 \end{vmatrix} e_3e_2e_1. \end{aligned}$$

But, if we consider the coefficients of $e_2e_3e_4$, &c., we see at once that they are the four coordinates of the plane through xyz, and we may therefore say that *the product of three points is the plane through them.*

If we take four points, we get

$$xyzw = \begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix} e_1e_2e_3e_4.$$

If we take five points, we find that their product vanishes identically.

We have just seen that a plane X can be written in the form

$$X = X_1e_2e_3e_4 + X_2e_3e_1e_4 + X_3e_1e_2e_4 + X_4e_3e_2e_1 \ldots\ldots\ldots\ldots (\Theta),$$

where $e_2e_3e_4$, &c. are the faces of the tetrahedron of reference; or, denoting these by E_1, E_2, E_3, E_4, we have

$$X = X_1E_1 + X_2E_2 + X_3E_3 + X_4E_4.$$

But now there is a difficulty. If we take two planes given in the form (Θ), and try to multiply them together, we shall fail, for the product will vanish identically, since each term will necessarily contain at least

one square factor; but the principle of duality obviously requires that the product of two planes shall be their line of intersection.

Grassmann obviates this difficulty by what he calls *regressive* multiplication. Before we can define this, we must introduce another conception of the greatest importance, viz., the *Ergänzung* (what I call the *conjugate*) of a quantity.

Grassmann supposes that e_1, e_2, e_3, e_4 are so chosen that

$$e_1 e_2 e_3 e_4 = 1,$$

or, if we please, we may say that this product is taken as the unit of such products.

Now this equation can be written

$$e_1 (e_2 e_3 e_4) = 1,$$

and $e_2 e_3 e_4$ is defined as the *conjugate* of e_1, and denoted by Ke_1,* and, in the same way, if E is any product of not more than four units, KE is defined by

$$E . KE = 1.$$

Thus $Ke_1 e_2 = e_3 e_4$, $Ke_1 e_2 e_3 = +e_4$, $Ke_4 = -e_1 e_2 e_3$, &c.: it will be seen hereafter that this term *conjugate* is appropriate. We have, if X is a plane,

$$KX = -(X_1 e_1 + X_2 e_2 + X_3 e_3 + X_4 e_4).$$

The definition of *regressive* multiplication is as follows:†—If E, E' are two different products of three units, EE' is defined by

$$K(EE') = KE . KE'.$$

Thus, if $E = e_1 e_2 e_3$, $E' = e_1 e_2 e_4$, we have $KE = e_4$, $KE' = -e_3$,

$$K(EE') = -e_4 e_3 = e_3 e_4,$$

$$EE' = e_1 e_2.$$

Using this definition of regressive multiplication, we can verify that *the product of two planes is their line of intersection;* that *the product of three planes is their point of intersection;* that *the product of four planes is the determinant of their coordinates;* and, lastly, that *the product of five planes vanishes identically.*

We can also verify that the product of a point and a plane vanishes if the plane contains the point; that the product of a point and a line is the plane through the point and line, and vanishes if the line contains the point; and that the product of a plane and a line is the point of intersection of the plane and line, and vanishes if the line is in the plane.

We must now consider the multiplication of lines. We saw, by

* Grassmann denotes it by $[e_1]$; my notation is, of course, borrowed from Hamilton.

† For the space of three dimensions we are considering.

multiplying two points, that the expression for a line was of the form

$$a = a_1e_1e_4 + a_2e_2e_4 + a_3e_3e_4 + a_4e_2e_3 + a_5e_3e_1 + a_6e_1e_2,$$

and it follows, from the expressions for a_1, &c., that we have

$$a_1a_4 + a_2a_5 + a_3a_6 = 0.*$$

Now take two lines a, b, and multiply them together: we get, as is easily verified,

$$ab = a_1b_4 + a_2b_5 + a_3b_6 + b_1a_4 + b_2a_5 + b_3a_6,\dagger$$

and we thus see that the equation of condition satisfied by the coordinates of a line a may be written

$$a^2 = 0.$$

But it is known that

$$a_1b_4 + a_2b_5 + a_3b_6 + a_4b_1 + a_5b_2 + a_6b_3 = 0$$

is the condition that the two lines whose coordinates are

$$(a_1a_2a_3a_4a_5a_6), \quad (b_1b_2b_3b_4b_5b_6),$$

respectively, may intersect. We may therefore say (using Hamilton's indispensable word *scalar*) that *the product of two lines is a scalar, which vanishes if the lines intersect.*

It may be useful to give a few illustrations of these processes. Suppose we have three planes connected by a linear relation

$$a = \lambda b + \mu c;$$

to interpret this, let D be any point, then we have

$$aD = \lambda bD + \mu cD.$$

Therefore, if bD, cD both vanish, aD will also vanish; that is, any point in b and c is in a; that is, a, b, c pass through the same straight line.

In exactly the same way, we can show that, if a, b, c are points such that

$$a = \lambda b + \mu c,$$

the three points are collinear.

Now, suppose that a, b, c are three lines, connected by this linear relation; then, if P is any point, and p any plane, we have

$$aP = \lambda bP + \mu cP,$$

$$ap = \lambda bp + \mu cp.$$

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* In a more familiar notation, this is the condition $af + bg + ch = 0$.

† $af' + bg' + ch' + a'f + b'g + c'h$.

Therefore (1) the planes joining the lines to any point are collinear, therefore the three lines are concurrent; (2) the intersections of the lines with any plane are collinear, therefore the three lines are complanar.

Again, let a, b, c, d be four lines such that

$$a = \lambda b + \mu c + \nu d;$$

let α be any other line; then we have

$$a\alpha = \lambda b\alpha + \mu c\alpha + \nu d\alpha;$$

therefore $a\alpha$ vanishes if $b\alpha$, $c\alpha$, $d\alpha$ all vanish: that is, any line cutting b, c, d cuts a: that is, a, b, c, d are four generators of the same species of an hyperboloid of one sheet.

Lastly, suppose we have a homography,

$$\gamma = \lambda\alpha + \mu\beta,$$
$$\gamma' = \lambda\alpha' + \mu\beta',$$

on two lines $\alpha\beta$, $\alpha'\beta'$, then

$$\gamma\gamma' = \lambda^2\alpha\alpha' + \lambda\mu\,(\alpha\beta' - \alpha'\beta) + \mu^2\beta\beta';$$

therefore the connectors of corresponding points of two homographies generate an hyperboloid of one sheet; for, if we write down four of these equations we shall be able to eliminate $\alpha\alpha'$, $\alpha\beta' - \alpha'\beta, \beta\beta'$, so as to get a linear relation between four lines $\gamma\gamma'$; this is necessary because $\alpha\beta' - \alpha'\beta$ is not a line, as its square is $2\,(\alpha\beta\alpha'\beta')$.

We have now to extend our conceptions. An expression like $a_1e_1e_4 + \&c.$, in which every term contains a product of two units, will be called a *form of the second degree*. We have seen that such a form represents a straight line if its square vanishes; but, if its square does not vanish, it must obviously mean something. We can see what it means as follows: let a be any form of the second degree, and let x be a line,* then

$$ax = a_4x_1 + a_5x_2 + a_6x_3 + a_1x_4 + a_2x_5 + a_3x_6.$$

Therefore $ax = 0$ is a linear relation among the six coordinates of the line x; that is, it is the equation of a *linear complex*, and we may therefore say that *a form of the second degree whose square does not vanish represents a linear complex*. And it is worth while to notice that this is the only way in which we can get a definite notion of the meaning of a linear complex in three-dimensional space; the ordinary conceptions of it are either (1) as a locus of lines, which is too restricted, or (2) as a thing with six coordinates, that is, a point in five-

* Coordinates $(x_1x_2x_3x_4x_5x_6)$.

dimensional space, which is too wide. The definition I have just given is perfectly definite and precise, and enables us to see the meaning, in our space, of all operations on the linear complex.*

There are two other interpretations of the form of the second degree, which are of equal importance.

Now, consider a system of forces acting on a rigid body. It is a known theorem in mechanics that they may be reduced in one way to the six forces acting along the six edges of a tetrahedron. Now we know that the edge $\overline{e_1 e_4}$ is represented by the product $e_1 e_4$, and we see that a force acting along this edge can be represented by $a_1 e_1 e_4$; where a_1 is a scalar proportional to the intensity of the force. Considering the forces along all the edges, we see that any system of forces can be represented in the form

$$a = a_1 e_1 e_4 + a_2 e_2 e_4 + a_3 e_3 e_4 + a_4 e_2 e_3 + a_5 e_3 e_1 + a_6 e_1 e_2.$$

But this is the general form of the second degree: therefore *the general form of the second degree represents a system of forces acting on a rigid body.*

Now we know that a system of forces can be reduced to what Prof. Ball calls a *wrench* about a certain screw. We may therefore say that the *general form of the second degree represents a wrench about a certain screw.*

There is also a kinematical interpretation. Any motion of a body can be reduced to rotations about the six edges of a given tetrahedron; now we habitually represent a rotation about a line by a length along the line, and we therefore see that a rotation about the edge $\overline{e_1 e_4}$ can be represented by $a_1 e_1 e_4$. Proceeding as above, we see that *the general form of the second degree represents a motion of a rigid body;* or, again, introducing Prof. Ball's terminology, we can say that *the general form of the second degree represents a twist about a certain screw.* Now, considering the three interpretations we have found, and attending only to what is common to the mechanical and kinematical interpretations, we can say that *the general form of the second degree represents a linear complex or a screw.* That is, the screw and the linear complex are the same thing regarded from different points of view. The *Ausdehnungslehre* furnishes a complete explanation of the theory of screws and the theory of the linear complex, and shows that they are identical and not merely analogous; this is the view on which the following paper is based.

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* There is a paper by Prof. Reye in a recent number of *Crelle*, on systems of complexes, which is rendered much less clear than it need have been by the want of a precise definition of the linear complex as an element of space.

It remains to point out the connection with the non-Euclidean geometry.

So far, the tetrahedron of reference has been perfectly arbitrary; we now suppose it real and self-conjugate with respect to the absolute, then the point-equation to the absolute will be

$$x_1^2+x_2^2+x_3^2+x_4^2=0.$$

The plane-equation (tangential equation) will be

$$X_1^2+X_2^2+X_3^2+X_4^2=0.$$

The line equation (condition that a line may touch a surface)* will be

$$a_1^2+a_2^2+a_3^2+a_4^2+a_5^2+a_6^2=0;$$

Now, if we use these forms of the equations of the absolute, the ordinary expressions for the distance between two points become

$$\cos xy=\frac{x_1y_1+x_2y_2+x_3y_3+x_4y_4}{\sqrt{(x_1^2+x_2^2+x_3^2+x_4^2)}\,.\,\sqrt{(y_1^2+y_2^2+y_3^2+y_4^2)}},$$

$$\sin xy=+\frac{\sqrt{\left\{\begin{array}{c}(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2+(x_1y_2-x_2y_1)^2\\+(x_1y_4-x_4y_1)^2+(x_2y_4-x_4y_2)^2+(x_3y_4-x_4y_3)^2\end{array}\right\}}}{\sqrt{(x_1^2+x_2^2+x_3^2+x_4^2)}\,.\,\sqrt{(y_1^2+y_2^2+y_3^2+y_4^2)}}$$

Now, it is easy to verify that these expressions can be written

$$\cos xy=\frac{x\,.\,Ky}{\sqrt{(xKx)}\,.\,\sqrt{(yKy)}},$$

$$\sin xy=\frac{\sqrt{(xy\,.\,Kxy)}}{\sqrt{(xKx)}\,.\,\sqrt{(yKy)}}.$$

I now write Sxy for $x\,.\,Ky$, Tx for $+\sqrt{(xKx)}$.†

Then we see that the above expressions can be written

$$\cos xy=\frac{Sxy}{Tx\,Ty},\qquad \sin xy=\frac{T\,(xy)}{Tx\,Ty}.$$

We have, of course, precisely similar expressions for the sine and cosine of the angle between two planes, and to these can be added the expressions for the angle between two lines or two complexes used in the paper.]

The Theory of Screws in Elliptic Space has been considered by Clifford and by Professor Ball,‡ but I venture to hope that the

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* Clifford's *rank-equation*.

† This is not Grassmann's notation.

‡ *Clifford*—"Preliminary Sketch of Biquaternions," *Mathematical Papers*, pp. 180 to 200; also Fragments numbered xli., xlii., xliv.

Ball—"Certain Problems in the Dynamics of a Rigid System moving in Elliptic Space" (Trans. R.I.A., t. xxviii., pp. 159 to 187).

methods employed in the following paper will be found to justify its existence.

My special object is to show that the *Ausdehnungslehre* supplies all the necessary materials for a calculus of screws in elliptic space. Clifford was apparently led to construct his theory of biquaternions by the want of such a calculus; but Grassmann's method seems to afford a simpler and more natural means of expression than biquaternions: thus an expression that Clifford writes

$$\frac{\alpha-\beta+\omega V\alpha\beta}{1-S\alpha\beta},$$

would be written

$$\frac{\alpha\beta}{S\alpha\beta};$$

with the modifications of Grassmann's notation used in this paper: and the meaning of the expression seems more obvious if we use the second form.

Another result of the application of the *Ausdehnungslehre* is, that we are able to make use of certain definitions given by Grassmann, and, by so doing, to discuss the metric properties of elliptic space without explicitly introducing the absolute.

I.

The four points of reference are denoted by e_1, e_2, e_3, e_4. The "Ergänzung" of any quantity a I call its conjugate and denote by Ka. I write Sab for the product aKb, if a, b are of the same order, and I write Tx for the positive square root of Sxx.

The general expression for a form of the second order is

$$a = a_1e_1e_4+a_2e_2e_4+a_3e_3e_4+a_4e_2e_3+a_5e_3e_1+a_6e_1e_2,$$

A form of the second order represents a linear complex, or, what is the same thing, a screw or a motor, to use Clifford's expression.*

If a represents a line (*rotor*), it is the product of two forms of the first or third order, and we therefore have

$$0=\frac{a^2}{2}=a_1a_4+a_2a_5+a_3a_6.$$

If a, b are two complexes,

$$ab = a_1b_4+a_2b_5+a_3b_6+a_4b_1+a_5b_2+a_6b_3.$$

If ab vanishes, the complexes are said to be reciprocal: two lines are reciprocal if they cut.

* As a screw (motor) and a linear complex are exactly the same, I shall use the three words for the form of the second order according to the context.

Two quantities a, b determine two angles defined by the equations

$$\sin[ab] = \frac{Tab}{TaTb},$$

$$\cos(ab) = \frac{T(aKb)}{TaTb}.*$$

These angles are identical, unless a, b are both complexes (or lines) or one is a point (or plane) and the other a complex (not a line).

If a is a point on any line and b is a point on the conjugate line, Sab vanishes. Two lines are at right angles if either meets the conjugate of the other.

II.

Let a, b be any two lines: if a line c is to meet them both at right angles, it must satisfy the four equations

$$0 = ac = bc = Ka \,.\, c = Kb \,.\, c.$$

But obviously, if c satisfies these equations, Kc also satisfies them; but we know that in general there are only two lines satisfying the conditions, therefore we have the theorem: there are in general two lines meeting two given lines at right angles, and these two lines are at right angles.

If there are more than two lines meeting a, b at right angles, there is an infinite number, and a, b, Ka, Kb must be connected by a linear relation with scalar coefficients: in this case the lines a, b are said to be parallel.

If we have $$a = \lambda b + \mu Kb + \nu Ka,$$

we have also $$Ka = \lambda Kb + \mu b + \nu a.$$

Substituting,

$$a = b(\lambda + \mu\nu) + Kb(\mu + \lambda\nu) + a\nu^2 = 0;$$

but in general we shall not have a, b, Kb connected by a linear relation: for, if this were the case, they would be complanar and concurrent.

Therefore $$\nu^2 = 1, \quad \lambda + \mu\nu = 0.$$

That is, either $\nu = 1$, $\lambda = -\mu$, or $\nu = -1$, $\lambda = \mu$. The first pair gives

$$a - Ka = \lambda(b - Kb).$$

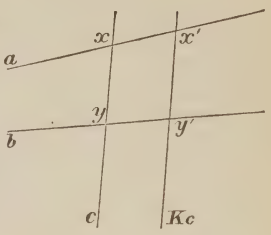
The second pair gives $$a + Ka = \lambda(b + Kb).$$

We thus see that there are two species of parallelism. Clifford distinguishes them as left and right parallelism respectively.

* It must be borne in mind that the tensor of a scalar is the scalar itself, whether it be positive or negative.

We shall now treat the subject of parallelism in a different way leading, however, to the same results.

Let xx', yy' be the points in which a, b are cut by c, Kc.



Then we have

$$0 = Sxx' = Sxy' = Syx' = Syy'.$$

Let $xy = \phi$, $x'y' = \phi'$.

Take any points $\xi = \alpha x + \alpha' x'$, $\eta = \beta y + \beta' y'$ on a, b respectively.

Then
$$\xi = \alpha\beta xy + \alpha\beta' xy' + \alpha'\beta x'y + \alpha'\beta' x'y',$$
$$S(xx'.\xi\eta) = \alpha\beta' T^2 x S x'y' - \alpha'\beta T^2 x' S xy.$$

Now we can suppose that the tensors of all points are unity: then $(\xi\eta)$ is at right angles to a if

$$\frac{\beta}{\beta'} = \frac{\alpha \cos\phi'}{\alpha' \cos\phi}.$$

But all our expressions are homogeneous, and we may therefore use this equation in the form

$$\beta = \alpha \cos\phi', \quad \beta' = \alpha' \cos\phi.$$

Now
$$\cos(\xi\eta) = \frac{\alpha\beta \cos\phi + \alpha'\beta' \cos\phi'}{(\alpha^2 + \alpha'^2)^{\frac{1}{2}} (\beta^2 + \beta'^2)}.$$

Substituting the values of β, β', we get

$$\cos(\xi\eta) = \cos\phi \cos\phi' . \frac{(\alpha^2 + \alpha'^2)^{\frac{1}{2}}}{(\alpha^2 \cos^2\phi' + \alpha'^2 \cos^2\phi)^{\frac{1}{2}}}.$$

This gives the perpendicular distance from b of any point in a. It follows (1) that ϕ, ϕ' are one the greatest and the other the least values of $\xi\eta$, and (2) that, if $\cos\phi = \pm \cos\phi'$, $\cos\xi\eta = \pm \cos\phi$.

If $\phi = \phi'$, a, b are said to be right parallel.

If $\phi + \phi' = \pi$, a, b are said to be left parallel.

We have now to express these conditions in terms of the coordinates of the lines.

We have
$$Sab = S(xx'.yy') = \begin{vmatrix} Sxy & Sxy' \\ Sx'y & Sx'y' \end{vmatrix}$$
$$= \cos\phi \cos\phi',$$

$$T^2 ab = S(xx'yy'.xx'yy').$$
$$= \begin{vmatrix} T^2x & 0 & Sxy & 0 \\ 0 & T^2x' & 0 & Sx'y' \\ Sxy & 0 & T^2y & 0 \\ 0 & Sx'y' & 0 & T^2y' \end{vmatrix}.$$

Therefore $$\sin^2[ab] = \sin^2\phi \sin^2\phi'.$$

But all angles are supposed less than π; we therefore have

$$\sin[ab] = \sin\phi \sin\phi'.$$

And we found $$\cos(ab) = \cos\phi \cos\phi'.$$

Therefore, if the lines are right parallel,

$$\sin[ab] + \cos(ab) = 1.$$

If they are left parallel,

$$\sin[ab] - \cos(ab) = 1.$$

Taking the first condition, we have

$$\frac{ab + aKb}{Ta\,Tb} = 1.$$

But $$(a+Ka)(b+Kb) = 2(ab+aKb),$$

and $$T(a+Ka)\,T(b+Kb) = 2Ta\,Tab.$$

Therefore $$\frac{(a+Ka)(b+Kb)}{T(a+Ka)\,T(b+Kb)} = 1.$$

Now it is easy to see that the coordinates of $(a+Ka)$ are of the form $(a_1a_2a_3a_1a_2a_3)$, and that those of $b+Kb$ are of the form $(b_1b_2b_3b_1b_2b_3)$.

Therefore $$\frac{a_1b_1 + a_2b_2 + a_3b_3}{(a_1^2+a_2^2+a_3^2)^{\frac{1}{2}}(b_1^2+b_2^2+b_3^2)^{\frac{1}{2}}} = 1.$$

Therefore $$0 = (b_2a_3 - a_3b_2)^2 + (b_3a_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2.$$

That is, if a, b are real,

$$a_1 : a_2 : a_3 = b_1 : b_2 : b_3.$$

That is, $$a + Ka = \lambda(b+Kb),$$

where λ is a scalar.

In precisely the same way, we get as the condition for left parallelism* $$a - Ka = \lambda(b - Kb).$$

III.

If a is a motor, (aa) vanishes and $[aa]$ is called the pitch of the motor. If a, b are two motors, $[ab]$ is called their moment. If a is a motor, and $a - Ka = 0$, a is called a right vector; if $a + Ka = 0$, a is called a left vector.

Any motor can be expressed in one way only, as the sum of a right and a left vector.

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* If we use the absolute, there is no difficulty in obtaining these conditions geometrically.

For, let a be the given motor, α, β the required vectors: we have

$$\alpha+\beta=a,$$

and therefore $$K\alpha+K\beta=Ka,$$

that is, $$\alpha-\beta=Ka.$$

Therefore $$\alpha=\tfrac{1}{2}(a+Ka),\quad \beta=\tfrac{1}{2}(a-Ka).$$

If a motor is reduced to the form $\lambda b+\mu Kb$, where b is a line and λ, μ scalars, b is called the axis of the motor.

To find the axis of a given motor.

We are to have $$a=\lambda b+\mu Kb,$$

and therefore $$Ka=\lambda Kb+\mu b.$$

Therefore $$b=\frac{\lambda a-\mu Ka}{\lambda^2-\mu^2}.$$

Since b can be multiplied by any scalar, we can take

$$b=\lambda a-\mu Ka,$$

and then $$a^2=2\lambda\mu\, T^2b,$$

$$T^2a=(\lambda^2+\mu^2)\,T^2b.$$

Therefore $$\frac{\lambda^2+\mu^2}{\lambda\mu}=\frac{2T^2a}{a^2},$$

that is, $$\frac{\lambda}{\mu}+\frac{\mu}{\lambda}=\frac{2}{\sin\phi},$$

if ϕ is the pitch of a.

A solution of this is $$\frac{\lambda}{\mu}=\cot\tfrac{1}{2}\phi\,;$$

and then we can take $\lambda=\cos\frac{1}{2}\phi$, $\mu=\sin\frac{1}{2}\phi$.

Therefore $$b=a\cos\tfrac{1}{2}\phi-Ka\sin\tfrac{1}{2}\phi.$$

The other solution $\lambda/\mu=\tan\frac{1}{2}\phi$ would simply change b and Kb into $-Kb$ and $-b$.

Now let α, α' be two motors: let β, β' be their axes and ϕ, ϕ' their pitches. Then

$$\beta=\alpha\cos\tfrac{1}{2}\phi-K\alpha\sin\tfrac{1}{2}\phi,$$

$$\beta'=\alpha'\cos\tfrac{1}{2}\phi'-K\alpha'\sin\tfrac{1}{2}\phi'.$$

Using these expressions to calculate $\beta\beta'$ and $S\beta\beta'$, and then solving for $\alpha\alpha'$, and remembering that (as is easily proved)

$$T\beta=T\alpha\cos\phi,\quad T\beta'=T\alpha'\cos\phi',$$

we get $$\sin[\alpha\alpha']=\sin[\beta\beta']\cos\tfrac{1}{2}(\phi-\phi')+\cos(\beta\beta')\sin\tfrac{1}{2}(\phi+\phi').$$

This equation corresponds to the expression

$$\Delta \sin\phi + (k+k')\cos\phi,$$

for the moment of two complexes in parabolic space.

IV.

The axes of two complexes α, β intersect at right angles if $S\alpha\beta=0$, $\alpha\beta=0$. Every congruence contains two complexes satisfying these conditions. Let the congruence be referred to these complexes, so that, if γ is any complex of the congruence, we have

$$\gamma = x\alpha + y\beta.$$

Let the pitches of α, β, γ be A, B, Γ; let $(\alpha\gamma)=a$, $(\beta\gamma)=b$. Then, remembering that $S\alpha\beta=\alpha\beta=0$, we get

$$\gamma^2 = x^2\alpha^2 + y^2\beta^2, \quad T^2\gamma = x^2T^2\alpha + y^2T^2\beta\ ;$$
$$S\alpha\gamma = xT^2\alpha, \quad S\beta\gamma = yT^2\beta.$$

The last two equations give

$$x = \frac{T\gamma\cos a}{T\alpha}, \qquad y = \frac{T\gamma\cos b}{T\beta},$$

$$\sin\Gamma = \frac{\gamma^2}{T^2\gamma} = \frac{x^2\alpha^2 + y^2\beta^2}{x^2T^2\alpha + y^2T^2\beta}$$
$$= \frac{\cos^2 a\sin A + \cos^2 b\sin B}{\cos^2 a + \cos^2 b}.$$

But
$$\cos^2 a + \cos^2 b = \frac{x^2T^2\alpha + y^2T^2\beta}{T^2\gamma} = 1.$$

Therefore
$$\sin\Gamma = \cos^2 a\sin A + \cos^2 b\sin B.$$

This answers to the expression

$$P = P_\alpha\cos^2 l + P_\beta\sin^2 l,$$

for the parameter of any complex of a congruence.*

V.

Since the condition that two screws may be reciprocal is linear in the coefficients of both, and since any screw can be expressed in terms of any six, the whole theory of reciprocal screws and of the screw set† remains true for elliptic space. It is also possible to extend the theory of the principal screws of inertia, if we make use of the following definitions and of the kinetic postulate involved in them:—

"Whatever may be the nature of the constraints acting on a rigid body, there is always a matrix a of the sixth order (called the funda-

* Or for the pitch of any screw on a cylindroid. (*Theory of Screws*, p. 15.)

† I venture to use this word for the screw *complex*, as the word *complex* is used in Plücker's sense in this paper.

mental matrix) such that, if an impulsive wrench be applied to the body about a screw A, the body will begin to turn about the screw aA, which is called the instantaneous screw corresponding to the impulsive screw A: aA means that the coordinates of A are transformed by the matrix a.

"The body is said to have freedom of the n^{th} order if it is possible to determine a screw set of order $(6-n)$, such that, if B be any screw of the set, $aB=0$. The set of the n^{th} order reciprocal to this set is called the set corresponding to the freedom of the n^{th} order possessed by the rigid body."

This being so, it follows at once that, if the body is free, there are six screws satisfying the equation $(a-\lambda)\,A=0$, and that, if referred to these six screws, the coordinates of an impulsive screw are $(x_1, x_2, x_3, x_4, x_5, x_6)$, the coordinates of the corresponding instantaneous screw are $(\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \lambda_4 x_4, \lambda_5 x_5, \lambda_6 x_6)$, where λ_1, &c. are the roots of $\mathrm{Det}\,(a-\lambda)=0$.

And then the whole theory in §§ 52 to 58 of the *Theory of Screws* can be at once extended to elliptic space.

VI.

It is worth while to consider the theory of the set of the third order. Let $\xi=\lambda\alpha+\mu\beta+\nu\gamma$ be any complex of the set: then ξ is a line if

$$(\alpha^2, \beta^2, \gamma^2, \beta\gamma, \gamma\alpha, \alpha\beta\,\mathcal{X}\lambda\mu\nu)^2=0.$$

Now we may take $(\lambda\mu\nu)$ as point coordinates in a plane; and then we may say that the points of the plane answer to complexes of a threefold set, and that the points of a certain conic answer to the lines of the set: two points, conjugate with respect to the conic, answer to reciprocal complexes. We can, in precisely the same way, represent the complexes of a fourfold set as points in space.

VII.

Clifford has applied biquaternions to the free motion of a solid in elliptic space.* We can also employ Grassmann's methods. If the velocity system of the body is given by a motor a, the rates of change of the coordinates of a point x are given by the equation

$$\dot{x}=K\,(ax);\dagger$$

and then, if T is the kinetic energy,

$$2T=\Sigma\,T^2\,(ax)\,dx$$
$$=A\,(a^2),$$

* *Mathematical Papers*, pp. 378 to 384.

† There seem to be some mistakes in sign on p. 382, but they do not affect the final result.

where A is a symmetrical matrix: and Aa^2 denotes

$$(A\,\Sigma a_1, a_2, a_3, a_4, a_5, a_6)^2.$$

The momentum of the body is given by

$$M = Aa.$$

And the equation of motion is $\dot{M} = 0$.

Clifford obtains the integral $T =$ constant; but two others can easily be found. It can be verified without difficulty that we have

$$\frac{1-a^2}{1-b^2-c^2}u^2 + \frac{1-b^2}{1-c^2-a^2}v^2 + \frac{1-c^2}{1-a^2-b^2} = \text{constant},$$

$$(1-a^2)a^2ux + (1-b^2)b^2vy + (1-c^2)c^2wz = \text{constant}.$$

VIII.

The whole theory admits of immediate extension to space of any odd number of dimensions. Let $2n-1$ be the number of dimensions: there is a form of order n which may be called a motor: the product of two motors is a scalar: if the square of a motor vanishes, it is called a rotor: the product of n forms of the first or $(2n-1)^{\text{th}}$ order is a rotor:* a motor has $\mu = \dfrac{2n!}{(n!)^2}$ coordinates: if the product of two motors vanishes, they are reciprocal: there are in general two rotors reciprocal to $(\mu-2)$ given motors: in particular there are two rotors reciprocal to $(\mu-2)/2$ given motors and their conjugates; if there is an infinite number, we have either

$$\Sigma\lambda\,(a+Ka) = 0 \quad \text{or} \quad \Sigma\lambda\,(a-Ka) = 0,$$

where the λ are scalars and the a are the given motors. The axis of a motor a is a rotor b, such that

$$a = \lambda b + \mu Kb.$$

It is obvious that the theory of screws can now be extended to space of $2n-1$ dimensions without any change in the notation or results.

IX.

It remains to explain the connection between the methods used in this paper and Clifford's Biquaternions: it is not possible to work out the correspondence between the two methods, any more than it is possible to work out in detail the correspondence between the *Ausdehnungslehre* and quaternions. The important point is, that Clifford's operator ω is the same as what is here denoted by K: if $\alpha = a_1 i + a_2 j + a_3 k$, $\beta = \beta_1 i + \beta_2 j + \beta_3 k$, the motor that Clifford writes

* The converse is not true.

$\alpha+\omega\beta$ would be written $a_1e_1e_4+a_2e_2e_4+a_3e_3e_4+\beta_1e_2e_3+\beta_2e_3e_1+\beta_3e_1e_2$ and denoted by a single symbol.

The difficulty in the application of biquaternions to parabolic space seems to arise from the definition of ω. Clifford's definition (p. 186) gives $\omega^2=0$: but there is nothing to prevent us from keeping to the definition $\omega=K$; that is to say, the symbol ω changes rotation about any axis into translation parallel to that axis, and translation parallel to any vector into rotation about a rotor equal to the vector and passing through a fixed point. The addition to Clifford's definition is contained in the words "and rotation . . . . a fixed point." With this definition we have $\omega^2=1$; we can introduce Clifford's operators

$$\xi=\frac{1+\omega}{2}, \quad \eta=\frac{1-\omega}{2},$$

and then we have $\xi^2=\xi$, $\eta^2=\eta$, $\xi\eta=0$; we can write any motor in the form $\xi\alpha+\eta\beta$, and we have

$$\frac{\xi\alpha+\eta\beta}{\xi\gamma+\eta\delta}=\xi q+\eta r;$$

where $q=\alpha/\gamma$, $r=\beta/\delta$.

Using the notations of my paper "On the Application of Quaternions to the Theory of the Linear Complex and the Linear Congruence,"* we can write the complex $(\alpha\beta)$ in the form $\alpha+\omega\beta$, and then $\omega(\alpha,\beta)=(\beta,\alpha)$.

[I think it right to state that, when the above paper was written, I had not seen Mr. Cox's paper in the *Camb. Phil. Trans.*, and only knew of it through a reference in Prof. Cayley's British Association address.—*June*, 1884.]

On Contact and Isolation, a Problem in Permutations.

By Mr. H. Fortey, M.A.

[*Read Jan. 10th*, 1884.]

1. Let there be n letters, and suppose a of them to be α, b of them to be β, c of them to be γ, and so on, where $n=a+b+c+$&c.; it is required to find the number of permutations of these n letters, taken all together, in which there are exactly r contacts of the α's, s contacts of the β's, t contacts of the γ's, and so on.

Def.—If a number of letters are ranged in a row, any two adjacent letters are *in contact*.

---

* *Messenger of Mathematics*, t. xii., p. 129.

2. In the first place we will determine the number of rows or permutations in which there are exactly r contacts of the a's, the distribution of the other letters being unrestricted.

It is clear that

if the a's occur in 1 group there are $a-1$ contacts,

„ 2 groups „ $a-2$ „

&c. &c.

Therefore, if the a's occur in $a-r$ groups there are r contacts.

Now let the a's be all written down in a row, thus :—

$$a \,.\, a \,.\, a \,.\, a \,.\, a \,.\, a \,.\, a \,.\, a \,.\, a$$

Then any $a-r-1$ vertical lines drawn through $a-r-1$ of the $a-1$ dots between the a's will divide the a's into a set of $a-r$ groups, and, if the lines be drawn in all possible ways, we shall have all the sets of $a-r$ groups in all orders of permutation.

But the lines can be drawn in ${}_{a-1}C_{a-r-1} = {}_{a-1}C_r$ ways; and this, therefore, is the number of permutations of sets of $a-r$ groups.

Now supposing that instead of ${}_{a-1}C_r$ sets of groups there were only *one* set, and the groups were all alike ($= A$ suppose); then, taking this set of $a-r$ A's with the other letters, we should have $n-r$ independent elements divided into sets of $a-r$, b, c, &c., and the number of permutations of these $n-r$ elements would be

$$\frac{\lfloor n-r}{\lfloor a-r \,\lfloor b \,\lfloor c \,\ldots}.$$

But instead of *one* set of $a-r$ elements there are, including permutations, ${}_{a-1}C_r$ sets.

Therefore the number of rows with $a-r$ groups in all possible positions is

$${}_{a-1}C_r \times \frac{\lfloor n-r}{\lfloor a-r \,\lfloor b \,\lfloor c \,\ldots} = P_r \text{ suppose.}$$

Here $${}_{a-1}C_r = \frac{(a-1)^{(r)}}{\lfloor r} \text{ and } \frac{1}{\lfloor a-r} = \frac{a^{(r)}}{\lfloor a}$$

(adopting factorial notation where $x^{(m)} = x\,(x-1)\,(x-2)\,\ldots$ to m factors); therefore

$$P_r = \frac{a^{(r)} \,.\, (a-1)^{(r)}}{\lfloor r} \,.\, \frac{\lfloor n-r}{\lfloor a \,\lfloor b \,\lfloor c \,\ldots} = \frac{a^{(r)} \,.\, (a-1)^{(r)}}{\lfloor r} \,.\, \frac{J^r \lfloor n}{\lfloor a \,\lfloor b \,\lfloor c \,\ldots},$$

where J is a convenient symbol due to Mr. Whitworth, such that

$$J^r \lfloor n = \lfloor n-r \,.$$

Now in P_r, or the number of rows with r contacts, P_{r+1} has been

added $r+1$ times. For in P_r the $a-r$ groups are in all possible positions, and therefore 2 groups may be in contact; but that will give a row of P_{r+1}. Conversely, take any row of P_{r+1}, then it may be supposed to arise from P_r with 2 groups in contact in any one of $r+1$ ways. Therefore P_{r+1} has been added $r+1$ times in P_r. But we require the number of rows with exactly r contacts, and therefore P_{r+1} should not have been added at all. Therefore we must subtract it $r+1$ times, which gives

$$P_r-(r+1)\,P_{r+1}.$$

But now it will be found that P_{r+2} has been added $\frac{(r+1)(r+2)}{2}$ times in P_r (since $r+2$ contacts can be broken in two places in $\frac{(r+1)(r+2)}{2}$ ways) and subtracted $(r+1)(r+2)$ times in $(r+1)\,P_{r+1}$; but it should have been neither added nor subtracted; therefore it must now be added $\frac{(r+1)(r+2)}{2}$ times, which gives

$$P_r-(r+1)\,P_{r+1}+\frac{(r+1)(r+2)}{2}\,P_{r+2};$$

and, pursuing this reasoning, we see that the number of rows with exactly r contacts of the α's is

$$P_r-(r+1)\,P_{r+1}+\frac{(r+1)(r+2)}{\lfloor 2}\,P_{r+2}-\&c.,\ \text{until the series stops,}$$

$$=\left\{\frac{a^{(r)}\,(a-1)^{(r)}}{\lfloor r}\,J^r-(r+1)\,\frac{a^{(r+1)}\,(a-1)^{(r+1)}}{\lfloor r+1}\,J^{r+1}\right.$$

$$\left.+\frac{(r+1)(r+2)}{\lfloor 2}\cdot\frac{a^{(r+2)}\,(a-1)^{(r+2)}}{\lfloor r+2}\,J^{r+2}-\&c.\right\}\frac{\lfloor n}{\lfloor a\ \lfloor b\ \lfloor c\ \ldots}$$

$$=\frac{a^{(r)}\,(a-1)^{(r)}}{\lfloor r}\left\{1-(a-r)(a-r-1)\,J\right.$$

$$\left.+\frac{(a-r)^{(2)}\,(a-r-1)^{(2)}}{\lfloor 2}\,J^2-\&c.\right\}\frac{J^r\,\lfloor n}{\lfloor a\ \lfloor b\ \lfloor c\ \ldots}$$

$$=\frac{{}^a\phi_r\,\lfloor n}{\lfloor a\ \lfloor b\ \lfloor c\ \ldots}\ \text{if}\ {}^a\phi_r=\frac{a^{(r)}\,(a-1)^{(r)}}{\lfloor r}\left\{1-(a-r)(a-r-1)\,J+\&c.\right\}J^r.$$

3. The operation ${}^a\phi_r$ is equivalent to the selection, from the whole number of permutations, of those in which there are exactly r contacts of the α's, the distribution of the other letters remaining altogether unrestricted. It is clear, therefore, that having operated with ${}^a\phi_r$ we can operate on the result with ${}^b\phi_s$, if we wish to ascertain the number of rows in which there are exactly r contacts of the α's and also s contacts of the β's.

Also $${}^{b}\phi_s\,{}^{a}\phi_r = {}^{a}\phi_r\,{}^{b}\phi_s,$$

for it is indifferent which selection is made first. And the same reasoning applies to any number of similar operations. Therefore the number of permutations with exactly r contacts of the α's, s contacts of the β's, t contacts of the γ's, and so on

$$= \frac{{}^{a}\phi_r\,{}^{b}\phi_s\,{}^{c}\phi_t \ldots \lfloor n}{\lfloor a \lfloor b \lfloor c \ldots}.$$

4. If the elements in any set a are unlike, then $\lfloor a$ should be struck out of the denominator in the foregoing formula; if the elements in *all* the sets are unlike, the formula becomes

$${}^{a}\phi_r\,{}^{b}\phi_s\,{}^{c}\phi_t \ldots \lfloor n.$$

5. If in a permutation no two elements of a particular set are in contact, the elements of that set may be said to be *isolated*. To find the number of rows in which the elements of a set a are isolated we must operate with ${}^{a}\phi_0$, and

$${}^{a}\phi_0 = 1 - a\,(a-1)\,J + \frac{a^{(2)}\,(a-1)^{(2)}}{\lfloor 2}\,J^2 - \&c.$$

6. Had my object been merely to solve the problem proposed in Art. 2, a shorter and more simple investigation than the one there given would have sufficed, and the result might have been expressed without the intervention of a series.

For, referring to that Article, we see that the number of sets of $a-r$ groups made up of a elements, including their permutations, is ${}_{a-1}C_r$.

Now write down the remaining $n-a$ elements in a row thus:—

$$. \times . \times . \times . \times . \times . \times . \times .$$

where the crosses represent the $n-a$ elements in any order whatever. Then the whole number of these rows is

$$\frac{\lfloor n-a}{\lfloor b \lfloor c \ldots}.$$

The number of dots in the line is $n-a+1$, and, from these, $a-r$ can be selected in ${}_{n-a+1}C_{a-r}$ ways.

Therefore the number of permutations with $a-r$ distinct groups of a's, or, in other words, with exactly r contacts of the a's,

$$= {}_{a-1}C_r \cdot {}_{n-a+1}C_{a-r} \cdot \frac{\lfloor n-a}{\lfloor b \lfloor c \ldots}$$

$$= \frac{a^{(r)}\,(a-1)^{(r)}}{\lfloor r} \cdot \frac{\lfloor n-a\,(n-a+1)^{(a-r)}}{\lfloor a \lfloor b \lfloor c \ldots}.$$

7. The result in this form is not susceptible of generalisation so as to include the case where the distribution of two or more sets of letters is restricted. It may however be transformed into the series given at the end of Art. 2, and I give the work as affording an independent proof of the truth of that formula.

$$(n-a+1)^{(a-r)} = (n-r-\overline{a-r-1})^{(a-r)} = E^{-\overline{a-r-1}}\,(n-r)^{(a-r)},$$

where E operates on n only.

Also $1+\Delta = E$, therefore $(1+\Delta)\,E^{-1}=1$, therefore $E^{-1}=1-\Delta E^{-1}$, therefore $(n-a+1)^{(a-r)} = (1-\Delta E^{-1})^{a-r-1}\,(n-r)^{(a-r)}$

$$= \left\{1-(a-r-1)\,\Delta E^{-1}+\frac{(a-r-1)^{(2)}}{\lfloor\underline{2}}\,\Delta^2 E^{-2}-\&c.\right\}(n-r)^{(a-r)}$$

$$= (n-r)^{(a-r)}-(a-r)(a-r-1)(n-r-1)^{(a-r-1)} + \frac{(a-r)^{(2)}\,(a-r-1)^{(2)}}{\lfloor\underline{2}}\,(n-r-2)^{(a-r-2)}-\&c.$$

Now the part involving n in the general term of this series is

$$(n-r-s)^{(a-r-s)},$$

and $(n-r-s)^{(a-r-s)} = (n-r-s)(n-r-s-1)\ldots(n-r-s-\overline{a-r-s}+1)$

$$= (n-r-s)(n-r-s-1)\ldots(n-a+1)$$

$$= \frac{\lfloor\underline{n-r-s}}{\lfloor\underline{n-a}},$$

therefore $\lfloor\underline{n-a}\,(n-a+1)^{(a-r)}$

$$= \lfloor\underline{n-r}-(a-r)(a-r-1)\lfloor\underline{n-r-1}+\frac{(a-r)^{(2)}(a-r-1)^{(2}}{\lfloor\underline{2}}\lfloor\underline{n-r-2}-\&c.$$

$$= \left\{1-(a-r)(a-r-1)\,J+\frac{(a-r)^{(2)}\,(a-r-1)^{(2)}}{\lfloor\underline{2}}\,J^2-\&c.\right\}J^r\lfloor\underline{n},$$

and substituting this value of $\lfloor\underline{n-a}\,(n-a+1)^{(a-r)}$ in the result in Art. 6 it becomes

$$\frac{a^{(r)}\,(a-1)^{(r)}}{\lfloor\underline{r}}\{1-(a-r)(a-r-1)\,J+\&c.\}\,\frac{J^r\lfloor\underline{n}}{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{c}\ldots} = \frac{{}^a\phi_r\lfloor\underline{n}}{\lfloor\underline{a}\,\lfloor\underline{b}\,\lfloor\underline{c}\ldots},$$

where ${}^a\phi_r$ is the operator already determined. Of the two proofs of the form of ${}^a\phi_r$ the last is perhaps the neatest, but the other has the advantage of showing the meaning of each term of the series.

Thursday, Feb. 14th, 1884.

Prof. HENRICI, F.R.S., President, and subsequently Sir J. COCKLE, F.R.S., Vice-President, in the Chair.

Messrs. D. Brockelbank and A. B. Basset were admitted into the Society.

The following communications were made:—

"On the Intersections of a Triangle with a Circle," Mr. H. M. Taylor.

"On the Function which denotes the Difference between the number of $(4m+1)$-divisors and the number of $(4m+3)$-divisors of a Number," Mr. J. W. L. Glaisher, F.R.S.

"On a General Theory including the Theories of Systems of Complexes and Spheres," Mr. A. Buchheim.

Prof. Sylvester, F.R.S., made some remarks on Matrices with reference to Nonions, &c.

The following presents were received:—

"Proceedings of the Royal Society," Vol. xxxv., No. 227; Vol. xxxvi., No. 228.

"Educational Times," for February.

"Johns Hopkins University Circulars," Vol. iii., No. 28.

"Compound Determinants," by C. A. Van Velzer, 4to Pamphlet.

"Beiblätter zu den Annalen der Physik und Chemie," B. viii., St. 1.

"Jahrbuch über die Fortschritte der Mathematik," xiii., 3 Jahrgang, 1881.

"Atti della R. Accademia dei Lincei," Vol. viii., F. 1.

"Nieuw Archief voor Wiskunde," Deel x., Stuck 2.

"Acta Mathematica," 3, 1.

"Journal für die reine und angewandte Mathematik," B. xcv., Heft 3 and 4; Berlin, 1884.

"Tidsskrift for Mathematik," V. Raekke, 1 Aargang, Hefte 1, 2, 3, 4, 5, 6; Kjøbenhavn, 1884.

Philosophical Society of Washington,—

"Alignment Curves on any Surface, with special Application to the Ellipsoid" (Abstract), by C. H. Kummell. (Read April 26, 1883.)

"The Theory of Errors practically Tested by Target-shooting" (Abstract), by the same Author.

On the Function which denotes the difference between the number of (4m+1)-divisors and the number of (4m+3)-divisors of a Number. By J. W. L. Glaisher, M.A., F.R.S.

[*Read Feb. 14th*, 1884.]

1. The excess of the number of the divisors of a number n which have the form $4m+1$ over the number of divisors which have the form $4m+3$, is a quantity which occurs in researches connected with the Theory of Numbers, and also as coefficient in certain systems of q-series in Elliptic Functions.

If we denote this quantity by $E(n)$, so that

$$E(n) = \text{number of divisors of } n \text{ of the form } \left.\begin{matrix} 4m+1 \\ - \quad\quad\quad\

The sign of each term in the developed expression is positive in the case of a divisor of the form $4m+1$, and negative in the case of a divisor of the form $4m+3$. Thus $E(n)$ is equal to the value of this product when a, b, c, ... are all replaced by unity; whence it follows that $E(n) = 0$, unless α, β, γ, ... are all even, in which case $E(n) = 1$. Next, suppose that $n = a^\alpha b^\beta c^\gamma \ldots r^\rho s^\sigma t^\tau \ldots$ where $a, b, c, \ldots$ are, as before, prime factors of the form $4m+3$, and where $r, s, t, \ldots$ are prime factors of the form $4m+1$. Then, reasoning as above, we see that $E(n)$ is equal to the value of

$$(1-a+a^2 \ldots \pm a^\alpha)(1-b+b^2 \ldots \pm b^\beta) \ldots$$
$$\times (1+r+r^2 \ldots + r^\rho)(1+s+s^2 \ldots + s^\sigma) \ldots,$$

when a, b, ... r, s, ... are all replaced by unity.

Denoting, as above, by $\phi(p)$ the number of divisors of p, we have therefore $E(n) = E(a^\alpha b^\beta c^\gamma \ldots) \times \phi(r^\rho s^\sigma t^\tau \ldots)$,

which, by means of the result found in the first case,

$$= \phi(r^\rho s^\sigma t^\tau \ldots) \text{ or } 0,$$

according as α, β, γ, ... are all even, or are not all even.

It has thus been shown that $E(n) = 0$ unless all the prime factors of n, which are of the form $4m+3$, occur with even exponents; in which case, if $n = 2^p u v^2$, all the prime factors of u being of the form $4m+1$, and all the prime factors of v of the form $4m+3$, then $E(n) = \phi(u)$. We see also that $E(n)$ cannot ever be negative.

3. It follows from the preceding investigation that, if $n = n_1 n_2 n_3 \ldots$, where n_1, n_2, n_3, ... are any relatively prime numbers, then

$$E(n) = E(n_1)\, E(n_2)\, E(n_3) \ldots$$

It is evident that, if p be a prime of the form $4m+1$, then

$$E(p^r) = r+1,$$

and that, if p be a prime of the form $4m+3$, then

$$E(p^r) = 1 \text{ or } 0,$$

according as r is even or uneven.

Also $$E(2^r) = 1.$$

By means of these formulæ we may write down at once the value of $E(n)$, when n has been resolved into its prime factors. For example, since $$495000 = 2^3 \times 3^2 \times 5^4 \times 11,$$

we have $$E(495000) = 1 \times 1 \times 5 \times 2 = 10.$$

4. The following table, which was calculated in the manner just explained, contains the values of $E(n)$ for all the values of n, up to $n = 1000$, for which $E(n)$ is not equal to zero.

TABLE OF THE VALUES OF $E(n)$ FROM $n = 1$ TO $n = 1000$.

n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$	n	$E(n)$
1	1	136	2	293	2	464	2	640	2	820	4
2	1	137	2	296	2	466	2	641	2	821	2
4	1	144	1	298	2	468	2	648	1	829	2
5	2	145	4	305	4	477	2	650	6	832	2
8	1	146	2	306	2	481	4	653	2	833	2
9	1	148	2	313	2	482	2	656	2	841	3
10	2	149	2	314	2	484	1	657	2	842	2
13	2	153	2	317	2	485	4	661	2	845	6
16	1	157	2	320	2	488	2	666	2	848	2
17	2	160	2	324	1	490	2	673	2	850	6
18	1	162	1	325	6	493	4	674	2	853	2
20	2	164	2	328	2	500	4	676	3	857	2
25	3	169	3	333	2	505	4	677	2	865	4
26	2	170	4	337	2	509	2	680	4	866	2
29	2	173	2	338	3	512	1	685	4	872	2
32	1	178	2	340	4	514	2	689	4	873	2
34	2	180	2	346	2	520	4	692	2	877	2
36	1	181	2	349	2	521	2	697	4	881	2
37	2	185	4	353	2	522	2	698	2	882	1
40	2	193	2	356	2	529	1	701	2	884	4
41	2	194	2	360	2	530	4	706	2	890	4
45	2	196	1	361	1	533	4	709	2	898	2
49	1	197	2	362	2	538	2	712	2	900	3
50	3	200	3	365	4	541	2	720	2	901	4
52	2	202	2	369	2	544	2	722	1	904	2
53	2	205	4	370	4	545	4	724	2	905	4
58	2	208	2	373	2	548	2	725	6	909	2
61	2	212	2	377	4	549	2	729	1	914	2
64	1	218	2	386	2	554	2	730	4	916	2
65	4	221	4	388	2	557	2	733	2	925	6
68	2	225	3	389	2	562	2	738	2	928	2
72	1	226	2	392	1	565	4	740	4	929	2
73	2	229	2	394	2	569	2	745	4	932	2
74	2	232	2	397	2	576	1	746	2	936	2
80	2	233	2	400	3	577	2	754	4	937	2
81	1	234	2	401	2	578	3	757	2	941	2
82	2	241	2	404	2	580	4	761	2	949	4
85	4	242	1	405	2	584	2	765	4	953	2
89	2	244	2	409	2	585	4	769	2	954	2
90	2	245	2	410	4	586	2	772	2	961	1
97	2	250	4	416	2	592	2	773	2	962	4
98	1	256	1	421	2	593	2	776	2	964	2
100	3	257	2	424	2	596	2	778	2	965	4
101	2	260	4	425	6	601	2	784	1	968	1
104	2	261	2	433	2	605	2	785	4	970	4
106	2	265	4	436	2	610	4	788	2	976	2
109	2	269	2	441	1	612	2	793	4	977	2
113	2	272	2	442	4	613	2	794	2	980	2
116	2	274	2	445	4	617	2	797	2	981	2
117	2	277	2	449	2	625	5	800	3	985	4
121	1	281	2	450	3	626	2	801	2	986	4
122	2	288	1	452	2	628	2	808	2	997	2
125	4	289	3	457	2	629	4	809	2		
128	1	290	4	458	2	634	2	810	2		
130	4	292	2	461	2	637	2	818	2		

The number of arguments for which $E(n)$ is not zero, and the sum of the values of $E(n)$ for each hundred numbers, are as follows :—

	Number of arguments.	Sum of values.
0—99	42	76
100—199	36	79
200—299	35	82
300—399	31	74
400—499	32	80
500—599	32	80
600—699	31	81
700—799	30	75
800—899	28	73
900—999	30	79
Total 0—999	327	779

5. The following two formulæ serve to express $E(n)$ linearly in terms of the E's of numbers less than n.

I.

If n be any uneven number, then

$$E(n)-2E(n-4)+2E(n-16)-2E(n-36)+\&c.$$
$$=0 \text{ or } (-1)^{\frac{1}{2}(\sqrt{n}-1)}\times\sqrt{n},$$

according as n is not, or is, a square number.

Every term in this formula is zero if n is of the form $4m+3$, so that no generality is lost by restricting n to the form $4m+1$.

II.

If n be any number,

$$E(n)-E(n-1)-E(n-3)+E(n-6)+E(n-10)-\&c.$$
$$=0 \text{ or } (-1)^n\times\tfrac{1}{4}\{(-1)^{\frac{1}{2}[\sqrt{(8n+1)}-1]}\times\sqrt{(8n+1)}-1\},$$

according as n is not, or is, a triangular number.

The numbers 1, 3, 6, 10, ..., which occur in the second formula, are the triangular numbers, given by the formula $\frac{1}{2}r(r+1)$, and, if n be itself a triangular number, the last term is $E(n-n)=0$. The signs of the terms after the first are negative and positive in pairs, the terms involving even triangular numbers in the argument having the positive sign, and those involving uneven triangular numbers having the negative sign.

Both formulæ are to be continued up to the term preceding the first term in which the argument becomes negative, *i.e.*, a term with negative argument is to be treated as zero. It may be noticed that,

if we *define* $E(n-n)$ to denote

$$(-1)^{n-1}\times\tfrac{1}{4}\{(-1)^{\frac{1}{2}[\sqrt{(8n+1)}-1]}\times\sqrt{(8n+1)}-1\},$$

then it follows, from II., that the expression

$$E(n)-E(n-1)-E(n-3)+E(n-6)+E(n-10)-\&c.$$

is equal to zero for *all* values of n; and this is perhaps the most convenient form in which to enunciate the theorem.

The following are examples of the formulæ. Putting $n=77$ and 81, the formula I. gives

$$E(77)-2E(73)+2E(61)-2E(41)+2E(13)=0,$$

and $E(81)-2E(77)+2E(65)-2E(45)+2E(17)=(-1)^{\frac{1}{2}(9-1)}\times 9;$

that is, $$0-2\times 2+2\times 2-2\times 2+2\times 2=0,$$

$$1-\ 0\ +2\times 4-2\times 2+2\times 2=9.$$

Putting $n=20$ and 21, the formula II. gives

$$E(20)-E(19)-E(17)+E(14)+E(10)-E(5)=0,$$

and $$E(21)-E(20)-E(18)+E(15)+E(11)-E(6)$$
$$=(-1)^{21}\times\tfrac{1}{4}\{(-1)^{6}\times 13-1\},$$

that is, $$2-0-2+0+2-2=0,$$

and $$0-2-1+0+0-0=-3.$$

6. The formulæ in the last section are of the same kind as Euler's celebrated formula

$$\psi(n)-\psi(n-1)-\psi(n-2)+\psi(n-5)+\psi(n-7)-\ldots=0,$$

where $\psi(n)$ denotes the sum of the divisors of n; the numbers 1, 2, 5, 7, ..., which occur in the arguments, are the pentagonal numbers given by the formula $\frac{1}{2}r(3r\pm 1)$; and $\psi(n-n)$ is defined to denote n. Euler regarded this formula as very remarkable, since it affords a means of calculating the sum of the divisors of n by a process in which none of the operations have any reference to the divisors, which remain unknown.*

7. The equations I. and II. were obtained by Elliptic Functions as follows:—

Denoting $\dfrac{2K}{\pi}$ by ρ, we have the following formulæ:

$$k^{\frac{1}{2}}\rho^{\frac{1}{2}}=2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+2q^{\frac{49}{4}}+\&c.,$$
$$k'^{\frac{1}{2}}\rho^{\frac{1}{2}}=1-2q+2q^{4}-2q^{9}+2q^{16}-\&c.,$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}}=2q^{\frac{1}{4}}-6q^{\frac{9}{4}}+10q^{\frac{25}{4}}-14q^{\frac{49}{4}}+\&c.,$$

* "Hoc autem merito eo mirabilius videtur, cum nulla operatio sit instituta, quæ ad rationem divisorum ullo modo referri queat; quin etiam divisores, quorum summa per hanc regulam reperitur, ipsi manent incogniti, etiamsi sæpe ex consideratione ipsius summæ concludi possint." – *Opera Arithmetica Collecta*, Vol. i., p. 150.

and also $$k^{\frac{1}{2}}\rho = \frac{2q^{\frac{1}{4}}}{1+q^{\frac{1}{2}}} + \frac{2q^{\frac{3}{4}}}{1+q^{\frac{3}{2}}} + \frac{2q^{\frac{5}{4}}}{1+q^{\frac{5}{2}}} + \frac{2q^{\frac{7}{4}}}{1+q^{\frac{7}{2}}} + \&c.$$
$$= 2\{E(1)\,q^{\frac{1}{4}} + E(5)\,q^{\frac{5}{4}} + E(9)\,q^{\frac{9}{4}} + E(13)\,q^{\frac{13}{4}} + \&c.\},$$

$$k'^{\frac{1}{2}}\rho = 1 - \frac{4q^2}{1+q^2} + \frac{4q^6}{1+q^6} - \frac{4q^{10}}{1+q^{10}} + \frac{4q^{14}}{1+q^{14}} - \&c.$$
$$= 1 - 4E(1)\,q^2 + 4E(2)\,q^4 - 4E(3)\,q^6 + 4E(4)\,q^8 - \&c.;$$

whence we deduce the identical equations

$$2\{E(1)\,q^{\frac{1}{4}} + E(5)\,q^{\frac{5}{4}} + E(9)\,q^{\frac{9}{4}} + \&c.\} = \frac{2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \&c.}{1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \&c.},$$

$$1 - 4E(1)\,q^2 + 4E(2)\,q^4 - 4E(3)\,q^6 + \&c. = \frac{2q^{\frac{1}{4}} - 6q^{\frac{9}{4}} + 10q^{\frac{25}{4}} - 14q^{\frac{49}{4}} + \&c.}{2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + 2q^{\frac{49}{4}} + \&c.}.$$

Dividing by $q^{\frac{1}{4}}$ in the first equation, and writing $q^{\frac{1}{2}}$ for q in the second, we have

$$E(1) + E(5)\,q + E(9)\,q^2 + \&c. = \frac{1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - \&c.}{1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \&c.} \quad \ldots \text{ (i.)},$$

$$1 - 4E(1)\,q + 4E(2)\,q^2 - 4E(3)\,q^3 + \&c. = \frac{1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \&c.}{1 + q + q^3 + q^6 + q^{10} + \&c.}$$
$$\ldots\ldots\ldots\ldots \text{(ii.)}$$

Thus $$\Sigma_0^\infty E(4n+1)\,q^n \times \Sigma_{-\infty}^{\infty} (-1)^n q^{n^2} = \Sigma_0^\infty (-1)^n (2n+1)\,q^{n(n+1)},$$
$$\{1 + 4\Sigma_1^\infty (-1)^n E(n)\,q^n\} \times \Sigma_0^\infty q^{\frac{1}{2}n(n+1)} = \Sigma_0^\infty (-1)^n (2n+1)\,q^{\frac{1}{2}n(n+1)}.$$

The formulæ I. and II. may be deduced from these results by equating the coefficients of q^n.

8. The following investigation is given here on account of the similarity of the resulting formula (which involves the difference between the sums of the even and uneven divisors of a number) to formula II. of § 5. We have

$$\frac{(1-q^2)(1-q^4)(1-q^6)\ldots}{(1-q)(1-q^3)(1-q^5)\ldots} = 1 + q + q^3 + q^6 + q^{10} + \&c.;$$

whence, taking the logarithm and differentiating,

$$\frac{q}{1-q} + \frac{3q^3}{1-q^3} + \frac{5q^5}{1-q^5} + \&c. - \frac{2q^2}{1-q^2} - \frac{4q^4}{1-q^4} - \frac{6q^6}{1-q^6} - \&c.$$
$$= \frac{q + 3q^3 + 6q^6 + 10q^{10} + 15q^{15} + \&c.}{1 + q + q^3 + q^6 + q^{10} + q^{15} + \&c.}.$$

Now, if we denote by $\zeta(n)$ the excess of the sum of the uneven divisors of n over the sum of the even divisors [so that $\zeta(n) = \psi(n)$

when n is uneven, and $\zeta(n)$ is negative when n is even], then it is evident that the left-hand member of this equation

$$= \zeta(1)\,q+\zeta(2)\,q^2+\zeta(3)\,q^3+\zeta(4)\,q^4+\&c.,$$

and we therefore find

$$\zeta(1)\,q+\zeta(2)\,q^2+\zeta(3)\,q^3+\zeta(4)\,q^4+\&c. = \frac{q+3q^3+6q^6+10q^{10}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.}$$

Writing this equation in the form .....(iii.).

$$\Sigma_1^\infty \zeta(n)\,q^n \times \Sigma_0^\infty q^{\frac{1}{2}n(n+1)} = \Sigma_1^\infty \tfrac{1}{2}n(n+1)\,q^{\frac{1}{2}n(n+1)},$$

and equating the coefficients of q^n, we find that

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\zeta(n-10)+\&c. = 0 \text{ or } n,$$

according as n is not, or is, a triangular number.

If, therefore, we define $\zeta(n-n)$ to denote $-n$, we have

$$\zeta(n)+\zeta(n-1)+\zeta(n-3)+\zeta(n-6)+\zeta(n-10)+\&c. = 0$$

for all values of n.

As examples of the formula, putting $n = 20$ and 21, as in Formula II., § 5, we find

$$\zeta(20)+\zeta(19)+\zeta(17)+\zeta(14)+\zeta(10)+\zeta(5) = 0,$$
$$\zeta(21)+\zeta(20)+\zeta(18)+\zeta(15)+\zeta(11)+\zeta(6) = 21;$$

that is

$$-30+20+18-8-6+6 = 0,$$
$$32-30-13+24+12-4 = 21.$$

9. The formula (ii.) may be expressed in a form corresponding to (iii.); viz., we have

$$E(1)\,q-E(2)\,q^2+E(3)\,q^3-E(4)\,q^4+\&c. = \frac{q-q^3+2q^6-2q^{10}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.}$$

.....(iv.),

$$\zeta(1)\,q+\zeta(2)\,q^2+\zeta(3)\,q^3+\zeta(4)\,q^4+\&c. = \frac{q+3q^3+6q^6+10q^{10}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.}$$

the numerator in the first equation being .....(iii.),

$$q-q^3+2q^6-2q^{10}+3q^{15}-3q^{21}+4q^{28}-4q^{36}+\&c.$$

This expression is evidently divisible by $1-q^2$, and, when divided by this factor, the quotient is

$$q+2q^6+2q^8+3q^{15}+3q^{17}+3q^{19}+4q^{28}+4q^{30}+4q^{32}+4q^{34}+5q^{45}+\&c.,$$

where the law of the terms is that, if $t_1, t_2, t_3, t_4, \ldots$ denote the triangular numbers 1, 3, 6, 10, ..., then the only exponents that occur are the numbers

$$t_{2r-1},\quad t_{2r-1}+2,\ t_{2r-1}+4,\ \ldots\ t_{2r-1}+2r-2$$

(*i.e.*, the r even numbers intermediate to $t_{2r-1}-2$ and t_{2r} if t_{2r-1} and t_{2r} are even, and the r uneven numbers intermediate to these limits when t_{2r-1} and t_{2r} are uneven), and the coefficient of each of these terms is r.

10. Replacing $\frac{1}{1-q^2}$ by $1+q^2+q^4+q^6+$&c., we thus find

$$\{E(1)q-E(2)q^2+E(3)q^3-E(4)q^4+\&c.\}\times\{1+q^2+q^4+q^6+q^8+\&c.\}$$
$$=\frac{q+2q^6+2q^8+3q^{15}+3q^{17}+3q^{19}+\&c.}{1+q+q^3+q^6+q^{10}+q^{15}+\&c.}.$$

Now, let $S(2n-1)=E(1)+E(3)+E(5)\ldots+E(2n-1)$,

$$S(2n)=E(2)+E(4)+E(6)\ldots+E(2n);$$

then the left-hand member of this equation is

$$S(1)q-S(2)q^2+S(3)q^3-S(4)q^4+\&c.,$$

and therefore

$$S(1)q-S(2)q^2+S(3)q^3-S(4)q^4+\&c.=\frac{q+2q^6+2q^8+3q^{15}+\&c.}{1+q+q^3+q^6+q^{10}+\&c.}$$
.....(v.)

Equating the coefficients of q^n in the equation

$$\Sigma_1^\infty(-1)^{n-1}S(n)q^n\times\Sigma_0^\infty q^{\frac{1}{2}n(n+1)}=q+2q^6+2q^8+3q^{15}+\&c.,$$

we find that

$$S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\&c.$$
$$=0 \text{ or } (-1)^{n-1}r,$$

according as n is not, or is, one of the numbers

$$t_{2r-1},\quad t_{2r-1}+2,\quad \ldots,\quad t_{2r-1}+2r-2.$$

This theorem may also be enunciated in the following singular form:

Counting $S(0)$, when it occurs, as a term, though assigning to it the value zero, then

$$S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\&c.$$

is zero, (i.) if the number of terms is uneven, or (ii.) if the argument of the last term is uneven; but (iii.), if the number of terms is even and the argument of the last term is also even (0 being regarded as an even argument), then it is equal to

$$(-1)^{n-1}\times\tfrac{1}{2}\text{ number of terms.}$$

As examples of the three cases, let $n=90$, 98, and 99. The theorem gives

(i.)

$$S(90)-S(89)-S(87)+S(84)+S(80)-S(75)-S(69)$$
$$+S(62)+S(54)-S(45)-S(35)+S(24)+S(12) \qquad =0,$$

(ii.)

$$S(98)-S(97)-S(95)+S(92)+S(88)-S(83)-S(77)$$
$$+S(70)+S(62)-S(53)-S(43)+S(32)+S(20)-S(7) \;=0,$$

(iii.)

$$S(99)-S(98)-S(96)+S(93)+S(89)-S(84)-S(78)$$
$$+S(71)+S(63)-S(54)-S(44)+S(33)+S(21)-S(8) \;=7.$$

In (i.) the positive terms are $36+34+32+24+22+9+5=162$,
and the negative terms are $37+35+30+28+19+13 \;=162$.

In (ii.) the positive terms are $37+36+34+27+24+12+9=179$,
and the negative terms are $39+37+31+30+22+17+3=179$.

In (iii.) the positive terms are $39+37+37+28+24+13+8=186$,
and the negative terms are $37+36+34+30+22+17+3=179$;

the difference being 7, as it should be.

The formula affords a complete verification of the accuracy of the values of a table of $E(n)$, for it involves all the arguments less than any given number n, and in such a manner that all the even-argument terms have one and the same sign, and all the uneven-argument terms have one and the same sign. Whenever, therefore, a term $E(r)$ enters, it occurs with the same sign, and an error in it would produce an increased effect (and could not be neutralised) by its repeated occurrence in the S-terms.

This will appear also from the developed form of the S-expression given in the next section.

11. The expression

$$(-1)^{n-1}\{S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\ldots\}$$

is identically equal to the coefficient of q^n in the product

$$\{E(1)\,q-E(2)\,q^2+E(3)\,q^3-\&c.\}$$
$$\times\{1+q^2+q^4+q^6+q^8+q^{10}+q^{12}+\&c.\}$$
$$\times\{1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\&c.\}.$$

Now we find by multiplication

$$(1+q^2+q^4+q^6+q^8+\&c.)\times(1+q+q^3+q^6+q^{10}+\&c.)$$
$$=$$
$$1$$
$$+q+q^2$$
$$+2q^3+q^4+2q^5$$
$$+2q^6+2q^7+2q^8+2q^9$$
$$+3q^{10}+2q^{11}+3q^{12}+2q^{13}+3q^{14}$$
$$+3q^{15}+\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

where the coefficients are

1
1, 1
2, 1, 2
2, 2, 2, 2
3, 2, 3, 2, 3
3, 3, 3, 3, 3, 3
4, 3, 4, 3, 4, 3, 4
4, 4, 4, 4, 4, 4, 4, 4
... ...

The successive groups of terms contain one, two, three, four, &c. members. The groups containing an even number of members consist of *repetitions ; e.g.*, the group of four consists of 4 twos, the group of six of 6 threes, &c. ; the groups containing an uneven number of members consist of *alternations*, *e.g.*, the group of three is 2, 1, 2 ; the group of five is 3, 2, 3, 2, 3 ; and, in general, in a group containing $2r$ members, each coefficient is equal to r, and in a group containing $2r+1$ members, the coefficients are

$$r+1,\ r,\ r+1,\ \ldots\ r,\ r+1\ ;$$

i.e., they are $r+1$ and r alternately, the first and last being $r+1$.

It follows, therefore, that

$$S(n)-S(n-1)-S(n-3)+S(n-6)+S(n-10)-\ldots$$

is identically equal to

$$\begin{gathered}
E(n) \\
-E(n-1)+E(n-2) \\
-2E(n-3)+E(n-4)-2E(n-5) \\
+2E(n-6)-2E(n-7)+2E(n-8)-2E(n-9) \\
+3E(n-10)-\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\
\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots \\
\ldots\quad \ldots\quad \ldots\quad \ldots\quad \ldots\quad \pm rE(1).
\end{gathered}$$

We notice that no term in this expression has a zero coefficient, which is in accordance with the remark made at the end of the last section.

12. Since the triple product considered in the preceding section is equal to $q+2q^6+2q^8+3q^{15}+3q^{17}+3q^{19}+\&c.$,

we find, by equating coefficients, the value of the E-expression for any given value of n, and we thus obtain a theorem which may be enunciated in the following curious form :

If n be any number, then

$$\begin{gathered}
E(n) \\
-E(n-1)+E(n-2) \\
-2E(n-3)+E(n-4)-2E(n-5) \\
+2E(n-6)-2E(n-7)+2E(n-8)-2E(n-9) \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \pm sE(0)
\end{gathered}$$

is equal to zero, if we assign to $E(0)$ the value 0 or 1 in accordance with the following rule: (i.) if $sE(0)$ is a term of an alternation, or (ii.) if $sE(0)$ is a term of a sequence and n and s are one even and the other uneven, then $E(0) = 0$; (iii.) if $sE(0)$ is a term of a sequence and n and s are both even or both uneven, then $E(0) = 1$.

For example, putting $n = 5, 6, 7$, we have

$$\begin{gathered}
(\alpha) \\
E(5) \\
-E(4)+E(3) \\
-2E(2)+E(1)-2E(0) = 0,
\end{gathered}$$

$$\begin{gathered}
(\beta) \\
E(6) \\
-E(5)+E(4) \\
-2E(3)+E(2)-2E(1) \\
+2E(0) \qquad\qquad = 0,
\end{gathered}$$

$$\begin{gathered}
(\gamma) \\
E(7) \\
-E(6)+E(5) \\
-2E(4)+E(3)-2E(2) \\
+2E(1)+2E(0) \qquad\qquad = 0.
\end{gathered}$$

In (α), $E(0)$ falls in an alternation, so that, by (i.), we put $E(0) = 0$; in (β), $E(0)$ falls in a sequence and 6 and 2 are both even, therefore, by (iii.), we put $E(0) = 1$; in (γ), $E(0)$ falls in a sequence, but 7 and 2 are one even and the other uneven, whence, by (ii.), we put $E(0) = 0$.

Substituting for the E's their values, the equations (α), (β), (γ) become

$$\begin{aligned}
(\alpha) \ldots\ldots\ & 2-1+0-2+1-0 \qquad\qquad = 0, \\
(\beta) \ldots\ldots\ & 0-2+1-0+1-2+2 \qquad = 0, \\
(\gamma) \ldots\ldots\ & 0-0+2-2+0-2+2+0 = 0.
\end{aligned}$$

The theorem contained in this section affords a complete verification of a table of $E(n)$, and it also serves to express $E(n)$ in terms of the E's of *all* the numbers inferior to n.

Other formulæ of the same class, but in which the coefficients are

regulated by a more simple law, will be investigated in the next two sections.

13. It was shown in § 7 that

$$E(1)+E(5)\,q+E(9)\,q^2+E(13)q^3+\&c.$$
$$=\frac{1-3q^2+5q^6-7q^{12}+9q^{20}-11q^{30}+13q^{42}-\&c.}{1-2q+2q^4-2q^9+2q^{16}-2q^{25}+2q^{36}-\&c.}.$$

If we divide the numerator of this expression by $1-q$, we obtain as quotient the expression

$$\begin{array}{c}1+q\\ -2q^2-2q^3-2q^4-2q^5\\ +3q^6+3q^7+3q^8+3q^9+3q^{10}+3q^{11}\\ -4q^{12}-4q^{13}-4q^{14}-4q^{15}-4q^{16}-4q^{17}-4q^{18}-4q^{19}\\ +5q^{20}+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\end{array}$$

where the successive groups of two, four, six, ... terms have opposite signs, and each term in a group of $2r$ terms has the coefficient r.

If we divide the denominator by $1-q$, we obtain as quotient the expression

$$\begin{array}{c}1\\ -q-q^2-q^3\\ +q^4+q^5+q^6+q^7+q^8\\ -q^9-q^{10}-q^{11}-q^{12}-q^{13}-q^{14}-q^{15}\\ +q^{16}+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\end{array}$$

where the successive groups of one, three, five, ... terms have opposite signs, and all the terms have the coefficient unity.

We thus find

$$E(1)+E(5)\,q+E(9)\,q^2+E(13)\,q^3+\&c.$$
$$=\frac{1+q-2\,(q^2+q^3+q^4+q^5)+3\,(q^6+q^7+q^8+q^9+q^{10}+q^{11})-\&c.}{1-q-q^2-q^3+q^4+q^5+q^6+q^7+q^8-q^9-q^{10}-q^{11}-q^{12}-q^{13}-\&c.},$$

and, by equating coefficients, we may deduce the following theorem:

If p denote any number of the form $4m+1$, then

$$\begin{array}{c}E(p)\\ -E(p-4)-E(p-8)-E(p-12)\\ +E(p-16)+E(p-20)+E(p-24)+E(p-28)+E(p-32)\\ -E(p-36)-\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\pm E(1)\end{array}$$
$$=(-1)^{\frac{1}{2}(r-1)}\times\tfrac{1}{2}(r+1),$$

where r^2 is the greatest uneven square which does not exceed p. Thus, if p is itself a square, $r=\sqrt{p}$.

For example, let $p = 29$; the theorem gives

$$\begin{aligned} & E(29) \\ & -E(25)-E(21)-E(17) \\ & +E(13)+E(9)+E(5)+E(1) \\ & = (-1)^{\frac{1}{2}(5-1)} \times \tfrac{1}{2}(5+1), \end{aligned}$$

that is, $$2-3-0-2+2+1+2+1 = 3.$$

The table in § 4 was verified by means of this formula, the values of the positive and negative terms in

$$\begin{aligned} & E(997) \\ & -E(993)-E(989)-E(985) \\ & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \quad -E(5)-E(1) \end{aligned}$$

being found to be 188 and 204 respectively, so that the value of this expression is -16. The greatest uneven square not exceeding 997 is 31^2 and $(-1)^{\frac{1}{2}(31-1)} \times \frac{1}{2}(31+1) = -16$.

The preceding formula affords a simple and complete verification of a table of $E(n)$; for the terms are combined by mere addition and subtraction, and all the arguments are of the form $4m+1$. As $E(n)$ vanishes when n is of the form $4m+3$, the formula is thus free from the presence of a number of terms which must necessarily be zero. Also, the absence of the even-argument terms is not a disadvantage, for we may regard the definition of $E(n)$ as applying primarily to the case of n uneven, the extension to n even being made by means of the formula $E(2^m p) = E(p)$. In actually verifying a table of $E(n)$, the accuracy of the even-argument terms is completely proved by merely verifying that, for every even argument $2m$, $E(2m) = E(m)$.

14. From (iv.) of § 9 we have

$$E(1)q-E(2)q^2+E(3)q^3-\&c. = \frac{q(1-q^2)+2q^6(1-q^4)+3q^{15}(1-q^6)+\&c.}{1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+q^{28}+\&c.}$$

and, dividing both numerator and denominator by $1-q$, we find

$$E(1)q-E(2)q^2+E(3)q^3-\&c. = \frac{q+q^2+2q^6+2q^7+2q^8+2q^9+3q^{15}+\&c.}{1+2q+2q^2+3q^3+3q^4+3q^5+4q^6+\&c.},$$

the numerator being

$$\begin{aligned} & q+q^2 \\ & +2q^6+2q^7+2q^8+2q^9 \\ & +3q^{15}+3q^{16}+3q^{17}+3q^{18}+3q^{19}+3q^{20} \\ & +4q^{28}+ \ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

and the denominator being 1

$$+2q+2q^2$$
$$+3q^3+3q^4+3q^5$$
$$+4q^6+4q^7+4q^8+4q^9$$
$$+5q^{10}+\ \dots\quad\dots\quad\dots$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots$$

We may thus, by equating coefficients, obtain the theorem :

If n be any number, then

$$E(n)$$
$$-2E(n-1)+2E(n-2)$$
$$-3E(n-3)+3E(n-4)-3E(n-5)$$
$$+4E(n-6)-4E(n-7)+4E(n-8)-4E(n-9)$$
$$+5E(n-10)-\ \dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots\quad\dots$$
$$\dots\quad\dots\quad\dots\quad\dots\quad+(-1)^{n-1}rE(1)$$
$$=(-1)^{n-1}\times\tfrac{1}{2}s \text{ or } 0,$$

according as s is even or uneven, where s is what the coefficient of $E(0)$ would be, if the formula were continued one term further. Thus $s=r$, unless $rE(1)$ is the last term in a group, in which case $s=r+1$.

Taking as examples $n=5$ and 6, the formula gives

$$E(5)$$
$$-2E(4)+2E(3)$$
$$-3E(2)+3E(1)\qquad\qquad=0,$$

and

$$E(6)$$
$$-2E(5)+2E(4)$$
$$-3E(3)+3E(2)-3E(1)=(-1)^5\times 2,$$

since the coefficient of $E(0)$, if the formula were continued one term further, would be 4.

Substituting for the E's their values, these equations become

$$2-2+0-3+3\qquad=0,$$
$$0-4+2-0+3-3=-2.$$

15. If we divide by $1-q$ (*i.e.*, multiply by $1+q+q^2+q^3+$&c.) the numerator and denominator of the fraction in equation (iii.) of § 9,

viz., $$\frac{q+3q^3+6q^6+10q^{10}+15q^{15}+\&c.}{1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\&c.},$$

we find

$$\zeta(1)q+\zeta(2)q^2+\zeta(3)q^3+\&c.=\frac{q+q^2+4q^3+4q^4+4q^5+10q^6+10q^7+\&c.}{1+2q+2q^2+3q^3+3q^4+3q^5+4q^6+\&c.},$$

and, by equating coefficients as before, we obtain the theorem :

If n be any number, then

$$\begin{aligned}
&\zeta(n)\\
&+2\zeta(n-1)+2\zeta(n-2)\\
&+3\zeta(n-3)+3\zeta(n-4)+3\zeta(n-5)\\
&+4\zeta(n-6)+\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ +r\zeta(1)\\
&=\tfrac{1}{6}(s^3-s),
\end{aligned}$$

where s is the coefficient of $\zeta(0)$ if the series be continued one term further. As before, $s=r$ unless $r\zeta(1)$ is the last term of a group, in which case $s=r+1$. The right-hand member of the equation

$$=\tfrac{1}{6}(s+1)\,s\,(s-1),$$

and is therefore clearly integral, since $s-1$, s, and $s+1$ are consecutive numbers.

Putting, as before, $n=5$ and 6, the formula gives

$$\begin{aligned}
&\zeta(5)\\
&+2\zeta(4)+2\zeta(3)\\
&+3\zeta(2)+3\zeta(1)\qquad\qquad=\tfrac{1}{6}(3^3-3),
\end{aligned}$$

and

$$\begin{aligned}
&\zeta(6)\\
&+2\zeta(5)+2\zeta(4)\\
&+3\zeta(3)+3\zeta(2)+3\zeta(1)=\tfrac{1}{6}(4^3-4);
\end{aligned}$$

that is,

$$6+2\times-5+2\times4+3\times-1+3=4,$$

$$-4+2\times6+2\times-5+3\times4+3\times-1+3\times1=10.$$

16. It may be mentioned that a formula,* similar to the E-formula and the ζ-formula which have just been given, exists also in the case of the function ψ, where $\psi(n)$ denotes the sum of the divisors of n; viz., we have, if n be any number,

$$\begin{aligned}
&\psi(n)\\
&-2\psi(n-1)-2\psi(n-2)\\
&+3\psi(n-3)+3\psi(n-4)+3\psi(n-5)\\
&-4\psi(n-6)-\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\\
&\dots\ \dots\ \dots\ \dots\ +(-1)^{r-1}r\psi(1)\\
&=(-1)^s\times\tfrac{1}{6}(s^3-s),
\end{aligned}$$

where s has the same meaning as in the two preceding sections.

* This formula was communicated to the Cambridge Philosophical Society in February, 1884. The paper in which it occurs is in course of publication in the Society's *Transactions*.

Putting $n = 5$ and 6, the formula gives

$$\begin{aligned} &\psi(5) \\ &-2\psi(4)-2\psi(3) \\ &+3\psi(2)+3\psi(1) \qquad\qquad = -\tfrac{1}{6}(3^3-3), \end{aligned}$$

and

$$\begin{aligned} &\psi(6) \\ &-2\psi(5)-2\psi(4) \\ &+3\psi(3)+3\psi(2)+3\psi(1) = \tfrac{1}{6}(4^3-4), \end{aligned}$$

that is,

$$6 - 2\times 7 - 2\times 4 + 3\times 3 + 3 = -4,$$

$$12 - 2\times 6 - 2\times 7 + 3\times 4 + 3\times 3 + 3 = 10.$$

17. The formulæ contained in the three preceding sections may be enunciated in the following convenient form by including the term involving $E(0)$.

If n be any number, then I.

$$\begin{aligned} &E(n) \\ &-2E(n-1)+2E(n-2) \\ &-3E(n-3)+3E(n-4)-3E(n-5) \\ &+4E(n-6)- \ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad +(-1)^n sE(0) \qquad\qquad = 0, \end{aligned}$$

where $E(0)$ denotes $\frac{1}{2}$ or 0 according as s is even or uneven.

II.

$$\begin{aligned} &\zeta(n) \\ &+2\zeta(n-1)+2\zeta(n-2) \\ &+3\zeta(n-3)+3\zeta(n-4)+3\zeta(n-5) \\ &+4\zeta(n-6)+ \ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad +s\zeta(0) \qquad\qquad = 0, \end{aligned}$$

where $\zeta(0)$ denotes $-\frac{1}{6}(s^2-1)$.

III.

$$\begin{aligned} &\psi(n) \\ &-2\psi(n-1)-2\psi(n-2) \\ &+3\psi(n-3)+3\psi(n-4)+3\psi(n-5) \\ &-4\psi(n-6)- \ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad +(-1)^{s-1}s\psi(0) \qquad\qquad = 0, \end{aligned}$$

where $\psi(0)$ denotes $\frac{1}{6}(s^2-1)$.

Expressions for E (n), &c., as determinants.

18. If $P_1x - P_2x^2 + P_3x^3 - \&c. = \dfrac{b_1x+b_2x^2+b_3x^3+b_4x^4+\&c.}{1+a_1x+a_2x^2+a_3x^3+a_4x^4+\&c.}$,

then it can be shown that

$$P_n = \begin{vmatrix} b_1, & b_2, & b_3, & b_4, & b_5, & \ldots \\ 1, & a_1, & a_2, & a_3, & a_4, & \ldots \\ 0, & 1, & a_1, & a_2, & a_3, & \ldots \\ 0, & 0, & 1, & a_1, & a_2, & \ldots \\ 0, & 0, & 0, & 1, & a_1, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \end{vmatrix} (n \text{ rows}),$$

where the elements of the third row are the same as those of the second, but shifted one place to the right, the first element being a cipher; the elements of the fourth row are the same as those of the third, but shifted one place to the right, with two ciphers prefixed; and so on. Excepting only the first row, all the elements in any diagonal parallel to the principal diagonal are the same. In writing determinants of this form, it is sufficient therefore to give the first two rows.

By means of this theorem we may deduce at once from the formulæ given in this paper the following expressions for $E(n)$, &c., in which, for brevity, only the first two rows of the determinants are written:

(i.)

$$E(n) =$$

$$\begin{vmatrix} 1, 0, -1, 0, 0, 2, 0, 0, 0, -2, 0, 0, 0, 0, 3, 0, 0, 0, 0, 0, -3, \ldots \\ 1, 1, \quad 0, 1, 0, 0, 1, 0, 0, \quad 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, \quad 0, \ldots \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 1, 1, 0, 0, 0, 2, 2, 2, 2, 0, 0, 0, 0, 0, 3, 3, 3, 3, 3, 3, 0, 0, 0, 0, 0, \ldots \\ 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, \ldots \end{vmatrix}$$

$$=$$

$$\begin{vmatrix} 1, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 3, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, \ldots \\ 1, 1, 1, 2, 1, 2, 2, 2, 2, 2, 3, 2, 3, 2, 3, 3, 3, 3, 3, 3, 3, 4, 3, 4, 3, \ldots \end{vmatrix},$$

each determinant containing n rows.

(ii.)

$$E(4n+1) =$$

$$\begin{vmatrix} 1, 0, -3, 0, 0, 0, 5, 0, 0, 0, 0, 0, -7, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, \ldots \\ 1, 2, \quad 0, 0, 2, 0, 0, 0, 0, 2, 0, 0, \quad 0, 0, 0, 0, 2, 0, 0, 0, 0, 0, 0, \ldots \end{vmatrix}$$

$$=$$

$$(-)^n \begin{vmatrix} 1, \quad 1, -2, -2, -2, -2, 3, 3, 3, \quad 3, \quad 3, \quad 3, -4, -4, \ldots \\ 1, -1, -1, -1, \quad 1, \quad 1, 1, 1, 1, -1, -1, -1, -1, -1, \ldots \end{vmatrix},$$

each determinant containing $n+1$ rows.

(iii.)

$$S(n) = \left| \begin{matrix} 1, 0, 0, 0, 0, 2, 0, 2, 0, 0, 0, 0, 0, 0, 3, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, \ldots \\ 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, \ldots \end{matrix} \right|,$$

the determinant containing n rows.

(iv.)

$$(-)^{n-1}\zeta(n) = \left| \begin{matrix} 1, 0, 3, 0, 0, 6, 0, 0, 0, 10, 0, 0, 0, 0, 15, 0, 0, 0, 0, 0, 21, 0, 0, 0, \ldots \\ 1, 1, 0, 1, 0, 0, 1, 0, 0, \ 0, 1, 0, 0, 0, \ 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, \ldots \end{matrix} \right|$$

$$= \left| \begin{matrix} 1, 1, 4, 4, 4, 10, 10, 10, 10, 20, 20, 20, 20, 20, 35, 35, 35, 35, 35, \ldots \\ 1, 2, 2, 3, 3, \ 3, \ 4, \ 4, \ 4, \ 4, \ 5, \ 5, \ 5, \ 5, \ 5, \ 6, \ 6, \ 6, \ 6, \ldots \end{matrix} \right|,$$

each determinant containing n rows.

It may be added that

$$(-)^{n-1}\psi(n) = \left| \begin{matrix} 1, \ 1, -4, -4, -4, 10, \ 10, \ 10, \ 10, -20, -20, -20, -20, \ldots \\ 1, -2, -2, \ 3, \ 3, \ 3, -4, -4, -4, \ -4, \ 5, \ 5, \ 5, \ldots \end{matrix} \right|,$$

the determinant containing n rows.

These determinant-values are, of course, quite inappropriate for purposes of calculation, being at best but inconvenient forms of expressing the results given by the formulæ I. and II. of § 5, and the similar formulæ in §§ 8—16. They seem, however, worth notice, as affording, though in an impracticable form, definite numerical expressions for $E(n)$, &c.

The functions $E(n)$, $\chi(n)$, $\lambda(n)$.

19.* If n be uneven, the number of primary complex numbers having n as their norm is equal to $E(n)$. In the *Quarterly Journal of Mathematics* for June, 1884, Vol. xx., pp. 97—167, I have considered the function $\chi(n)$, which denotes the sum of the primary complex numbers having n as norm, and also $\lambda(n)$ the sum of their squares. These functions are connected with one another and with $E(n)$ and $\psi(n)$ by a number of relations in which the terms follow laws similar to those which occur in this paper. The following two formulæ are perhaps the most curious of these relations; they would afford complete verifications of a table giving the values of $E(n)$ and $\chi(n)$.

* This section has been added since the paper was read.

If p be any number of the form $4m+1$, then

(i.)

$$E(p)$$
$$-2E(p-4)-2E(p-8)$$
$$+3E(p-12)+3E(p-16)+3E(p-20)$$
$$-4E(p-24)-\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$=$$
$$\chi(p)$$
$$+2\chi(p-4)+2\chi(p-8)$$
$$+3\chi(p-12)+3\chi(p-16)+3\chi(p-20)$$
$$+4\chi(p-24)-\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

(ii.)

$$E(p)$$
$$-2E(p-8)-2E(p-16)$$
$$+3E(p-24)+3E(p-32)+3E(p-40)$$
$$-4E(p-48)-\ \ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$=$$
$$(-)^n\{\chi(p)$$
$$+2\chi(p-8)+2\chi(p-16)$$
$$+3\chi(p-24)+3\chi(p-32)+3\chi(p-40)$$
$$+4\chi(p-28)+\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\},$$

the series being continued so long as the arguments remain positive.

The Relations of the Intersections of a Circle with a Triangle.

By Mr. H. M. Taylor.

[*Read Feb. 14th*, 1884.]

If, in a given triangle ABC, a triangle $\alpha\beta\gamma$ be inscribed, and the circumscribed circle of the triangle $\alpha\beta\gamma$ cut the sides BC, CA, AB again respectively in α', β', γ', then, as will be proved, if the triangle $\alpha\beta\gamma$ remains constant in shape (see Fig. 1)—

(1) The angles of the triangle $\alpha'\beta'\gamma'$ are determinate.

(2) The lines $\alpha\beta', \beta\gamma', \gamma\alpha', \alpha'\beta, \beta'\gamma, \gamma'\alpha$ remain constant in direction.

(3) If the lines $\beta\gamma', \gamma\alpha', \alpha\beta'$ (see Fig. 2) form a triangle $A'B'C'$, and

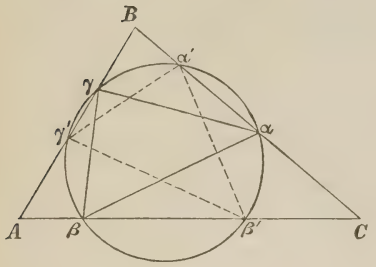

FIG. 1.

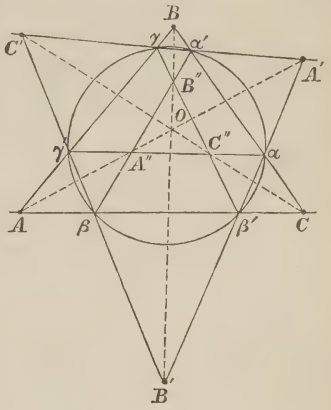

FIG. 2.

the lines $\gamma\beta', \alpha\gamma', \beta\alpha'$ form a triangle $A''B''C''$, the triangles ABC, $A'B'C'$, $A''B''C''$ are copolar, their common pole O being a fixed point.

(4) The ratio of the radii of the circles ABC, $\alpha\beta\gamma$ is proportional to the sine of the inclination of one of the sides of the triangle $\alpha\beta\gamma$ to a fixed straight line, *i.e.*, $\frac{R}{r} = L\sin(\psi+\theta)$.

(5) The locus of the centre of the circle $\alpha\beta\gamma$ is a straight line.

(6) The envelope of any side of the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ is a parabola which touches two sides of the triangle ABC.

(7) The circle $\alpha\beta\gamma$ envelopes a conic whose centre is at the centre of the minimum $(\alpha\beta\gamma)$ circle, and one of whose axes is in the direction of the locus of the centre of the $(\alpha\beta\gamma)$ circle.

1. *The angles of the triangle $\alpha'\beta'\gamma'$ are determinate.*

In Fig. 1, all the six points lie in the sides BC, CA, AB of the triangle ABC.

In this case, which may be regarded as the typical case, the relations which exist between the angles of the triangles ABC, $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, may be written

$$\left.\begin{aligned} \alpha+\alpha' &= B+C \\ \beta+\beta' &= C+A \\ \gamma+\gamma' &= A+B \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This may be proved at once, as follows (see Fig. 1):

$$\beta = \alpha\beta\gamma = \pi - \alpha\alpha'\gamma = B\alpha'\gamma,$$
$$\beta' = \alpha'\beta'\gamma' = \pi - \alpha'\gamma\gamma' = B\gamma\alpha',$$

therefore $$\beta+\beta' = B\alpha'\gamma + B\gamma\alpha' = \pi - B = A + C.$$

Similarly it can be shewn that

$$\alpha+\alpha' = B+C,$$

and

$$\gamma+\gamma' = A+B.$$

We may select the pairs of triangles in *four* different ways, and the proposition applies equally to all the combinations.

The point γ may be taken with the following pairs:

$$\alpha, \beta; \quad \alpha, \beta'; \quad \alpha', \beta; \quad \alpha', \beta',$$

and it is easily seen that

$$\beta\alpha\gamma+\beta'\alpha'\gamma' = \beta'\alpha\gamma+\beta\alpha'\gamma' = \beta\alpha'\gamma+\beta'\alpha\gamma' = \beta'\alpha'\gamma+\beta\alpha\gamma',$$

the sum of each pair of angles being equal to the sum

$$\beta\alpha\gamma+\beta'\alpha\gamma'.$$

In Fig. 3, which may be regarded as the exceptional case, the circle cuts one side BC twice, but the other two sides produced. In this case

$$\left.\begin{aligned} \alpha+\alpha' &= \pi+A \\ \beta+\beta' &= B \\ \gamma+\gamma' &= C \end{aligned}\right\} \quad \ldots\ldots(2).$$

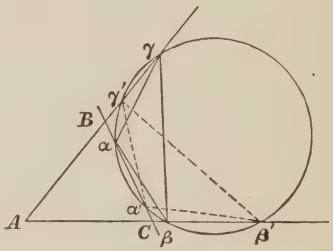

Fig. 3.

This may be proved thus (see Fig. 3):

$$\alpha = \beta\alpha\gamma = \pi-\beta\beta'\gamma,$$

and

$$\alpha' = \beta'\alpha'\gamma' = \pi-\beta'\gamma\gamma',$$

therefore

$$\alpha+\alpha' = \pi-A\beta'\gamma+\pi-A\gamma\beta' = \pi+A;$$

also

$$\beta = \alpha\beta\gamma = \pi-\alpha\gamma'\gamma = B\gamma'\alpha,$$

and

$$\beta' = \alpha'\beta'\gamma' = \pi-\alpha'\alpha\gamma' = B\alpha\gamma',$$

therefore

$$\beta+\beta' = B\gamma'\alpha+B\alpha\gamma' = ABC = B;$$

and similarly

$$\gamma+\gamma' = C.$$

The only special case of interest in the exceptional case (Fig. 3) is when the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$ become coincident, and the circle becomes an escribed circle of the triangle ABC. We shall therefore confine our attention for the future to the general case in which equations (1) hold.

Special Cases.

The most interesting special cases to consider are

Case I., where

$$\left.\begin{aligned} \alpha' &= \alpha \\ \beta' &= \beta \\ \gamma' &= \gamma \end{aligned}\right\} \quad \ldots\ldots(3),$$

whence
$$\left.\begin{aligned} \alpha &= \frac{\pi}{2} - \frac{A}{2} \\ \beta &= \frac{\pi}{2} - \frac{B}{2} \\ \gamma &= \frac{\pi}{2} - \frac{C}{2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Case II., where
$$\left.\begin{aligned} \alpha &= A \\ \beta &= B \\ \gamma &= C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

whence
$$\left.\begin{aligned} \alpha' &= \pi - 2A \\ \beta' &= \pi - 2B \\ \gamma' &= \pi - 2C \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Case III., where
$$\left.\begin{aligned} \alpha' &= \gamma \\ \beta' &= \alpha \\ \gamma' &= \beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

whence
$$\left.\begin{aligned} \alpha &= C = \beta' \\ \beta &= A = \gamma' \\ \gamma &= B = \alpha' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

If we apply the general case of equations (1) to the triangle $A'B'C'$ (Fig. 2), we obtain at once the equations

$$\left.\begin{aligned} \beta + \gamma' &= B' + C' \\ \gamma + \alpha' &= C' + A' \\ \alpha + \beta' &= A' + B' \end{aligned}\right\},$$

whence
$$\left.\begin{aligned} A' &= C + \gamma - \beta \\ B' &= A + \alpha - \gamma \\ C' &= B + \beta - \alpha \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9);$$

and, if we apply the equations (1) to the triangle $A''B''C''$, we obtain

$$\left.\begin{aligned} \gamma + \beta' &= B'' + C'' \\ \alpha + \gamma' &= C'' + A'' \\ \beta + \alpha' &= A'' + B'' \end{aligned}\right\},$$

whence
$$\left.\begin{aligned} A'' &= B + \beta - \gamma \\ B'' &= C + \gamma - \alpha \\ C'' &= A + \alpha - \beta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

2. *The lines* $\alpha\beta'$, *&c. remain constant in direction.*

It is seen at once, from Fig. 4, that the inclinations of $\alpha\beta$ to the sides CA, CB are γ, γ' respectively.

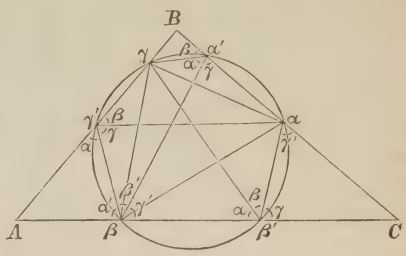

Fig. 4.

3. *The triangles* ABC, $A'B'C'$, $A''B''C''$ *are copolar, their common pole* O *being a fixed point.*

Now, applying Pascal's theorem to the points taken in the order $\alpha'\alpha\beta'\beta\gamma'\gamma$, we see that the pairs of lines $\alpha'\alpha$, $\beta\gamma'$; $\beta'\beta$, $\gamma\alpha'$; $\gamma'\gamma$, $\alpha\beta'$ intersect on a straight line ; or (Fig. 2) that the pairs of sides BC, $B'C'$; CA, $C'A'$; AB, $A'B'$ of the triangles ABC, $A'B'C'$ intersect on a straight line. In other words, the triangles ABC, $A'B'C'$ are coaxal. They are therefore copolar, *i.e.*, AA', BB', CC' meet in a point, O suppose.

Again, applying Pascal's theorem to the points taken in the order $\alpha\alpha'\beta\beta'\gamma\gamma'$, we see that the pairs of lines

$$\alpha\alpha',\ \beta'\gamma\ ;\quad \beta\beta',\ \gamma'\alpha\ ;\quad \gamma\gamma',\ \alpha'\beta$$

intersect on a straight line ; or (Fig. 2) that the pairs of sides BC, $B''C''$; CA, $C''A''$; AB, $A''B''$ of the triangles ABC, $A''B''C''$ intersect on a straight line.

In other words, the triangles ABC, $A''B''C''$ are coaxal. They are therefore copolar, *i.e.*, AA'', BB'', CC'' meet in a point.

Again, applying Pascal's theorem to the points taken in the order $\beta'\beta\alpha'\gamma\gamma'\alpha$, we see that the pairs of lines

$$\beta'\beta,\ \gamma\gamma'\ ;\quad \beta\alpha',\ \gamma'\alpha\ ;\quad \alpha'\gamma,\ \alpha\beta'$$

intersect on a straight line ; *i.e.*, (Fig. 2) $AA''A'$ is a straight line.

Similarly $BB''B'$, $CC''C'$ are straight lines.

Therefore we have three straight lines, viz., $AA''A'$, $BB''B'$, $CC''C'$, passing through the point O.

It may be remarked that, so far, this proposition is true for any conic section.

It appears, from Fig. 4, that as long as $\alpha\beta\gamma$ is of constant shape, the lines $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$ remain parallel to themselves ; and the same remark applies to the lines $\beta'\gamma$, $\gamma'\alpha$, $\alpha'\beta$.

We see therefore (Fig. 2) that, since $\gamma'\beta$, $\gamma'A''$, $\beta A''$ are all fixed in direction, $AA''A'$ is a fixed straight line.

Similarly, $BB''B'$, $CC''C'$ are fixed straight lines, and therefore O is a fixed point.

4. $\dfrac{R}{r} = L\sin(\psi+\theta)$.

As the whole figure is determinate, when we fix three of the six

points α, β, γ, α', β', γ', it is clear that we can express any angle and any line in the figure in terms of the sides and angles of the triangle ABC, of the angles α, β, γ, and of the inclination of one of the sides of the triangle $\alpha\beta\gamma$ to one of the sides of ABC.

If we call the angle $B\alpha\gamma$, θ, the inclinations of the other sides of $\alpha\beta\gamma$ to the corresponding sides of ABC are given in Fig. 5.

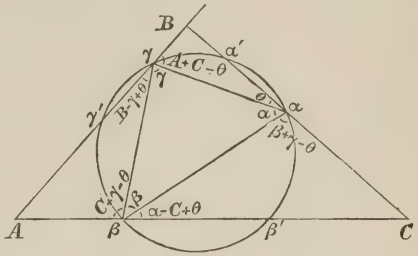

Fig. 5.

We will now establish the relation which exists between the angle θ, and the radius of the circumscribing circle of the triangle $\alpha\beta\gamma$, assuming the angles α, β, γ known.

We have (Fig. 5)

$$B\alpha = \gamma\alpha \frac{\sin B\gamma\alpha}{\sin \gamma B\alpha} = \gamma\alpha \frac{\sin (A+C-\theta)}{\sin B},$$

$$C\alpha = \beta\alpha \frac{\sin C\beta\alpha}{\sin \beta C\alpha} = \beta\alpha \frac{\sin (\alpha-C+\theta)}{\sin C},$$

and

$$B\alpha + C\alpha = BC = a.$$

Therefore, substituting for $B\alpha$, $C\alpha$, and using the relations

$$\gamma\alpha = 2r \sin \beta,$$

$$\beta\alpha = 2r \sin \gamma,$$

where r is the radius of the circle $\alpha\beta\gamma$, we obtain

$$2r \sin \beta \frac{\sin (A+C-\theta)}{\sin B} + 2r \sin \gamma \frac{\sin (\alpha-C+\theta)}{\sin C} = a.$$

Now, if R be the radius of the circumscribing circle of the triangle ABC, we have $a = 2R \sin A$. Therefore we may write

$$\frac{R}{r} = \frac{\sin \beta \sin (B+\theta)}{\sin A \sin B} + \frac{\sin \gamma \sin (\alpha-C+\theta)}{\sin A \sin C} \quad \ldots\ldots\ldots\ldots (11).$$

This equation may be written

$$\frac{R}{r} = L \sin (\psi+\theta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

where

$$\left.\begin{aligned} L \sin \psi &= \frac{\sin \beta}{\sin A} + \frac{\sin \gamma \sin (\alpha-C)}{\sin A \sin C} \\ L \cos \psi &= \frac{\sin \beta}{\sin A} \cot B + \frac{\sin \gamma \cos (\alpha-C)}{\sin A \sin C} \end{aligned}\right\}.$$

It is clear that L must be a symmetrical function of A and α, B and β, C and γ. We will proceed to find $L^2 \sin^2 A \sin^2 B \sin^2 C$.

From the last equations we have

$$L^2 \sin^2 A = \left(\sin\beta + \frac{\sin\gamma \sin(\alpha - C)}{\sin C}\right)^2 + \left(\sin\beta \cot B + \frac{\sin\gamma \cos(\alpha - C)}{\sin C}\right)^2$$

$$= \frac{\sin^2\beta}{\sin^2 B} + \frac{\sin^2\gamma}{\sin^2 C} + \frac{2\sin\beta\sin\gamma}{\sin B \sin C}\cos(B + C - \alpha),$$

therefore $L^2 \sin^2 A \sin^2 B \sin^2 C$

$$= \sin^2\beta \sin^2 C + \sin^2\gamma \sin^2 B - 2\sin\beta\sin\gamma\sin B\sin C\cos(A+\alpha)$$
$$= \sin^2\beta\sin^2 C + \sin^2\gamma\sin^2 B - 2\cos\alpha\sin\beta\sin\gamma\cos A\sin B\sin C$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C$$
$$= \sin^2\beta\sin C(\sin A\cos B + \cos A\sin B)$$
$$+ \sin^2\gamma\sin B(\sin A\cos C + \cos A\sin C)$$
$$- 2\cos\alpha\sin\beta\sin\gamma\cos A\sin B\sin C$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C$$
$$= \sin^2\beta\sin A\cos B\sin C + \sin^2\gamma\sin A\sin B\cos C$$
$$+ \cos A\sin B\sin C(\sin^2\beta + \sin^2\gamma - 2\cos\alpha\sin\beta\sin\gamma)$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C$$
$$= \sin^2\alpha\cos A\sin B\sin C + \sin^2\beta\sin A\cos B\sin C$$
$$+ \sin^2\gamma\sin A\sin B\cos C$$
$$+ 2\sin\alpha\sin\beta\sin\gamma\sin A\sin B\sin C \quad \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

5. *The locus of the centre of the circle $\alpha\beta\gamma$ is a straight line.*

It may be shewn in various ways that the locus of the centre of the circle $\alpha\beta\gamma$, when $\alpha\beta\gamma$ is of constant shape, is a straight line.

In Fig. 6 it is easily seen that, if x, y, z be the perpendiculars from

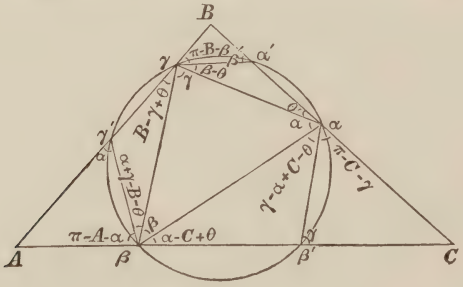

Fig. 6.

the centre of the circle $\alpha\beta\gamma$ on the sides of the triangle ABC, or, in other words, if x, y, z be the trilinear coordinates of the centre, using the triangle ABC as the triangle of reference, then

$$\left.\begin{aligned} x &= r\cos(\beta - \theta) \\ y &= r\cos(\gamma - \alpha + C - \theta) \\ z &= r\cos(\alpha + \gamma - B - \theta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

The coordinates x_1, y_1, z_1 of the minimum $(\alpha\beta\gamma)$ circle are found as follows :—

From equations (12) and (14),

$$\begin{aligned} x_1 &= \frac{R}{L}\cos\left(\beta+\psi-\frac{\pi}{2}\right) = \frac{R}{L}\sin(\beta+\psi) \\ &= \frac{R}{L^2}(L\sin\psi\cos\beta + L\cos\psi\sin\beta) \\ &= \frac{R}{L^2\sin A}\left\{\left(\sin\beta+\frac{\sin\gamma\sin(\alpha-C)}{\sin C}\right)\cos\beta \right. \\ &\qquad \left. +\left(\sin\beta\cot B+\frac{\sin\gamma\cos(\alpha-C)}{\sin C}\right)\sin\beta\right\} \\ &= \frac{R}{L^2\sin A}\left(\frac{\sin\beta\sin\beta'}{\sin B}+\frac{\sin\gamma\sin\gamma'}{\sin C}\right) \end{aligned}$$

similarly
$$\left.\begin{aligned} y_1 &= \frac{R}{L^2\sin B}\left(\frac{\sin\gamma\sin\gamma'}{\sin C}+\frac{\sin\alpha\sin\alpha'}{\sin A}\right) \\ z_1 &= \frac{R}{L^2\sin C}\left(\frac{\sin\alpha\sin\alpha'}{\sin A}+\frac{\sin\beta\sin\beta'}{\sin B}\right) \end{aligned}\right\} \quad \ldots\ldots(14a).$$

From these equations, by eliminating r and θ, we obtain the equation of the straight line, which is the locus of the centre, *i.e.*,

$$x\sin(A+2\alpha)+y\sin(B+2\beta)+z\sin(C+2\gamma)=0 \quad \ldots\ldots(15),$$

or
$$x\sin(\alpha-\alpha')+y\sin(\beta-\beta')+z\sin(\gamma-\gamma')=0.$$

We must observe, that Case I., *i.e.*, when $\alpha=A$, $\beta=B$, $\gamma=C$, is an exceptional one; for equations (14) then become

$$\left.\begin{aligned} x &= r\cos(B-\theta) \\ y &= r\cos(B-\theta) \\ z &= r\cos(B-\theta) \end{aligned}\right\}.$$

Hence we see that the centre is the point, whose coordinates satisfy the equations $x=y=z$, *i.e.*, the point is the centre of the inscribed circle of the triangle ABC.

In Case II., the equation (15) becomes

$$x\sin 3A+y\sin 3B+z\sin 3C=0 \quad \ldots\ldots(16),$$

which is satisfied by the coordinates of the centre of the nine-point circle of the triangle ABC, which is a particular case of Case II.

In Case III., the equation (15) becomes

$$x\sin(A+2C)+y\sin(B+2A)+z\sin(C+2B)=0,$$

which may be written

$$x\sin(B-C)+y\sin(C-A)+z\sin(A-B)=0 \quad \ldots\ldots(17),$$

which is satisfied by the coordinates of the centre of the circum-

scribing circle of the triangle ABC, which is a particular case of Case III.

If α, α' (see Fig. 2) coincide, the point A' will coincide with them. We will call this point α''.

It will be seen (see Fig. 5) that, when α, α' coincide, $\theta=\beta$, and that

$$B\alpha'' = a\gamma\,\frac{\sin\,(A+C-\beta)}{\sin B} = \frac{2r\sin\beta\sin\beta'}{\sin B},$$

$$C\alpha'' = a\beta'\,\frac{\sin\gamma}{\sin C} = \frac{2r\sin\gamma\sin\gamma'}{\sin C},$$

wherefore α'' divides the side BC in the ratio

$$\frac{\sin\beta\sin\beta'}{\sin B} : \frac{\sin\gamma\sin\gamma'}{\sin C}.$$

If β'', γ'' be the points where BO, CO cut the opposite sides, we shall have

$$\left.\begin{aligned} A\gamma'' : \gamma''B &= \frac{\sin\alpha\sin\alpha'}{\sin A} : \frac{\sin\beta\sin\beta'}{\sin B} \\ B\alpha'' : \alpha''C &= \frac{\sin\beta\sin\beta'}{\sin B} : \frac{\sin\gamma\sin\gamma'}{\sin C} \\ C\beta'' : \beta''A &= \frac{\sin\gamma\sin\gamma'}{\sin C} : \frac{\sin\alpha\sin\alpha'}{\sin A} \end{aligned}\right\} \quad \text{.....(18).}$$

It will be observed that the point O is the centre of mass of three masses proportional to

$$\frac{\sin A}{\sin\alpha\sin\alpha'},\quad \frac{\sin B}{\sin\beta\sin\beta''}\quad \frac{\sin C}{\sin\gamma\sin\gamma'},$$

placed at A, B, C respectively.

In Case I.,

$$B\alpha'' : \alpha''C = \frac{\sin^2\beta}{\sin B} : \frac{\sin^2\gamma}{\sin C} = \frac{\cos^2\frac{B}{2}}{\sin B} : \frac{\cos^2\frac{C}{2}}{\sin C}$$

$$= \cot\frac{B}{2} : \cot\frac{C}{2} = s-b : s-c \quad \text{.....(19)},$$

where $$2s = a+b+c;$$

therefore $$B\alpha'' = s-b,\quad C\alpha'' = s-c;$$

whence we see that α'' is the point of contact of the inscribed circle of ABC.

In Case III.,

$$B\alpha'' : \alpha''C = \sin\beta' : \sin\gamma' = \sin 2B : \sin 2C$$

$$= \sin B\cos B : \sin C\cos C \quad \text{.....(20)},$$

therefore $$B\alpha'' = a\,\frac{\sin 2B}{\sin 2B+\sin 2C} = \frac{a\sin B\cos B}{\sin A\cos\,(B-C)}.$$

In Case III.,

$$Ba'' : a''C = \frac{\sin A \sin C}{\sin B} : \frac{\sin B \sin A}{\sin C} = \sin^2 C : \sin^2 B \ldots\ldots (21),$$

therefore
$$Ba'' = a \frac{\sin^2 C}{\sin^2 C + \sin^2 B} = \frac{ac^2}{b^2+c^2},$$

$$Ca'' = \frac{ab^2}{b^2+c^2}.$$

6. *The envelope of $\beta\gamma$ is a parabola which touches AB, AC.*

We can shew that each side of the triangle $a\beta\gamma$, if it remain of constant shape, touches a parabola, which touches two sides of the triangle ABC.

Take the side $\beta\gamma$, and let us consider AC, AB as axes of oblique coordinates x and y.

Then the equation of $\beta\gamma$ may be written

$$\frac{x}{A\beta} + \frac{y}{A\gamma} = \frac{\beta\gamma}{\beta\gamma},$$

or (see Fig. 5),

$$\frac{x}{\sin(B-\gamma+\theta)} + \frac{y}{\sin(C+\gamma+\theta)} = \frac{2r\sin\alpha}{\sin A} = \frac{2R\sin\alpha}{L\sin A\sin(\psi-\theta)},$$

or
$$\frac{x}{\sin(\lambda+\theta)} + \frac{y}{\sin(\mu-\theta)} = \frac{M}{\sin(\psi-\theta)} \text{ suppose } \ldots\ldots\ldots (22),$$

$$2\sin(\psi-\theta)\{x\sin(\mu-\theta)+y\sin(\lambda+\theta)\} = 2M\sin(\lambda+\theta)\sin(\mu-\theta),$$

$$x\{\cos(\psi-\mu)-\cos(\psi+\mu-2\theta)\}+y\{\cos(\psi-\lambda-2\theta)-\cos(\psi+\lambda)\}$$
$$= M\{\cos(\lambda-\mu+2\theta)-\cos(\lambda+\mu)\},$$

which may be written

$$x\cos(\psi-\mu)-y\cos(\psi+\lambda)+M\cos(\lambda+\mu)$$
$$+\cos 2\theta\{-x\cos(\psi+\mu)+y\cos(\psi-\lambda)-M\cos(\lambda-\mu)\}$$
$$+\sin 2\theta\{-x\sin(\psi+\mu)+y\sin(\psi-\lambda)+M\sin(\lambda-\mu)\} = 0.$$

This we will write for the time

$$A+B\cos 2\theta + C\sin 2\theta = 0.$$

Therefore the envelope is

$$A^2 = B^2+C^2.$$

Therefore the equation of the envelope required is

$$\{x\cos(\psi-\mu)-y\cos(\psi+\lambda)+M\cos(\lambda+\mu)\}^2$$
$$= \{-x\cos(\psi+\mu)+y\cos(\psi-\lambda)-M\cos(\lambda-\mu)\}^2$$
$$+\{-x\sin(\psi+\mu)+y\sin(\psi-\lambda)+M\sin(\lambda-\mu)\}^2,$$

which, being rearranged, gives

$$x^2 \sin^2(\psi-\mu)+y^2\sin^2(\psi+\lambda)-2xy\sin(\psi-\mu)\sin(\psi+\lambda)$$
$$-2xM\sin(\psi-\mu)\sin(\lambda+\mu)$$
$$-2yM\sin(\psi+\lambda)\sin(\lambda+\mu)+M^2\sin^2(\lambda+\mu)=0 \quad \ldots\ldots\ldots(23).$$

This is the equation of a parabola which touches the axes of coordinates at

$$x_1 = M.\frac{\sin(\lambda+\mu)}{\sin(\psi-\mu)},$$

and at

$$y_1 = M.\frac{\sin(\lambda+\mu)}{\sin(\psi+\lambda)}.$$

Now $$\lambda = B-\gamma, \quad \mu = C+\gamma.$$

Therefore the coordinates of the points of contact are

$$x_1 = M.\frac{\sin A}{\sin(\psi-C-\gamma)}, \quad y_1 = M.\frac{\sin A}{\sin(\psi+B-\gamma)} \quad \ldots\ldots(24).$$

It will be seen that, since the inclination of $\beta\gamma$, $\beta'\gamma'$ to the sides AB, AC are merely interchanged, and their lengths are in the constant ratio $\frac{\sin\alpha}{\sin\alpha'}$, the corresponding enveloping parabolas will have the same relation.

In Case I., $$\psi = -\frac{B}{2}, \quad \gamma = \frac{\pi}{2}-\frac{C}{2},$$

therefore
$$\psi-C-\gamma = -\frac{B}{2}-C-\frac{\pi}{2}+\frac{C}{2} = \frac{A}{2}-\pi,$$
$$\psi+B-\gamma = -\frac{B}{2}+B-\frac{\pi}{2}+\frac{C}{2} = -\frac{A}{2}.$$

Therefore the distances x_1, y_1 of the points of contact with the sides are equal. Now

$$M\sin A = \frac{2R\sin\alpha}{L} = \frac{2R\cos\frac{A}{2}}{-\frac{R}{r_1}} = -2r_1\cos\frac{A}{2},$$

where r_1 is the radius of the inscribed circle of the triangle ABC.

Therefore
$$x_1 = y_1 = \frac{-2r_1\cos\frac{A}{2}}{-\sin\frac{A}{2}} = 2r_1\cot\frac{A}{2} = 2(s-a),$$

where $2s = a+b+c$.

The parabola therefore touches the sides AB, AC at double the distance from A of the points of contact of the inscribed circle of ABC.

In Case II., $$\psi = C+\frac{\pi}{2}, \quad \gamma = C,$$

therefore
$$\psi - C - \gamma = -C + \frac{\pi}{2}$$
$$\psi + B - \gamma = \quad B + \frac{\pi}{2},$$

and
$$M = \frac{2R \sin \alpha}{L \sin A} = R;$$

therefore
$$\left.\begin{aligned} x_1 &= R \frac{\sin A}{\cos C} = \frac{a}{2 \cos C} \\ y_1 &= R \frac{\sin A}{\cos B} = \frac{a}{2 \cos B} \end{aligned}\right\} \quad \text{.....(25).}$$

In Case III., $\psi = -\phi$, $L = -\dfrac{1}{\sin \phi}$, $\gamma = B$;

therefore
$$\psi - C - \gamma = -\phi - C - B = A - \phi - \pi,$$
$$\psi + B - \gamma = -\phi,$$
$$M = \frac{2R \sin \alpha}{L \sin A} = -\frac{2R \sin C \sin \phi}{\sin A} = -\frac{c \sin \phi}{\sin A},$$

therefore
$$\left.\begin{aligned} x_1 &= +\frac{c \sin \phi}{\sin (A - \phi)} = \frac{c \sin B \sin C}{\sin^2 A} \\ y_1 &= +\frac{c \sin \phi}{\sin \phi} = c \end{aligned}\right\} \quad \text{.....(26).}$$

Therefore the parabola touches AB at B and AC at a distance $b \dfrac{c^2}{a^2}$ from A.

7. *The envelope of the circle $\alpha\beta\gamma$ is a conic.*

Let x, y, z, x', y', z' be the coordinates of the centres of two equal circles of radius r; then, from equations (14),

$$x + x' = r \{\cos (\beta - \theta) + \cos (\beta - \theta')\} = 2r \cos \{\beta - \tfrac{1}{2} (\theta + \theta')\} \cos (\theta - \theta');$$

but, from equation (12),

$$\frac{R}{r} = L \sin (\psi + \theta) = L \sin (\psi + \theta'),$$

therefore
$$\theta + \psi = \pi - (\psi + \theta'),$$

or
$$\theta + \theta' = \pi - 2\psi = 2\theta_1 \text{ suppose,}$$

if θ_1 be the value of θ which makes the radius a minimum.

Therefore, if $\bar{x}$ be the coordinate of the middle point of the line joining the centres of the pair of equal circles (r),

$$\bar{x} = \tfrac{1}{2} (x + x') = r \cos (\beta - \theta_1) \cos (\theta - \theta'),$$

similarly
$$\bar{y} = \tfrac{1}{2} (y + y') = r \cos (\gamma - \alpha + C - \theta_1) \cos (\theta - \theta'),$$
$$\bar{z} = \tfrac{1}{2} (z + z') = r \cos (\alpha + \gamma - B - \theta_1) \cos (\theta - \theta').$$

Therefore, if x_1, y_1, z_1 be the coordinates of the centre of the minimum circle

$$\frac{\bar{x}}{x_1} = \frac{\bar{y}}{y_1} = \frac{\bar{z}}{z_1},$$

or the point $(\bar{x}, \bar{y}, \bar{z})$ coincides with the point (x_1, y_1, z_1).

If r_1 be the minimum radius,

$$\frac{R}{r_1} = L,$$

$$\frac{R}{r} = L \sin(\psi+\theta) = L \sin(\tfrac{1}{2}\pi-\theta_1+\theta) = L\cos(\theta_1-\theta),$$

therefore

$$r\cos(\theta-\theta_1) = r_1.$$

Now
$$\begin{aligned} x-x_1 &= r\cos(\beta-\theta) - r_1\cos(\beta-\theta_1) \\ &= r\{\cos(\beta-\theta) - \cos(\theta-\theta_1)\cos(\beta-\theta_1)\} \\ &= r\{\cos(\beta-\theta_1+\theta_1-\theta) - \cos(\theta-\theta_1)\cos(\beta-\theta_1)\} \\ &= r\,.\sin(\theta-\theta_1)\sin(\beta-\theta_1) = \lambda\,.\,\delta, \end{aligned}$$

where δ is the distance between the centres of the circles r, r_1.

Therefore, eliminating θ from the equations

$$\left.\begin{aligned} r\sin(\theta-\theta_1)\sin(\beta-\theta_1) &= \lambda\delta \\ r\cos(\theta-\theta_1) &= r_1 \end{aligned}\right\},$$

we obtain

$$r^2\sin^2(\beta-\theta_1) = \lambda^2\delta^2 + r_1^2\sin^2(\beta-\theta_1).$$

Now, if we take the centre of the circle r for the origin, and the locus of the centres of the $(\alpha\beta\gamma)$ circle as the axis of x, we may write the equation of the circle thus,

$$(x-\delta)^2 + y^2 = \lambda^2\delta^2\operatorname{cosec}^2(\beta-\theta_1) + r_1^2,$$

or

$$x^2+y^2-r_1^2-2\delta x = \delta^2\{\lambda^2\operatorname{cosec}^2(\beta-\theta_1) - 1\}.$$

The envelope therefore is

$$x^2\frac{\lambda^2}{\lambda^2-\sin^2(\beta-\theta_1)} + y^2 = r_1^2,$$

which is a conic section, whose axes are

$$r_1 \text{ and } r_1\sqrt{\left\{1-\frac{\sin^2(\beta-\theta_1)}{\sin^2\varphi}\right\}},$$

where φ is the angle which the straight line (15), the locus of the centre, makes with BC.

We will now examine the three special cases marked I., II., III. separately.

Case I.,

$$\left.\begin{aligned} \alpha &= \alpha' \\ \beta &= \beta' \\ \gamma &= \gamma' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This is the same as saying

$$\text{arc } \beta\gamma = \text{arc } \beta'\gamma' \text{ (see Fig. 7),}$$

&c.

therefore $$\text{arc } \beta\beta' = \text{arc } \gamma\gamma' = \text{arc } \alpha\alpha'.$$

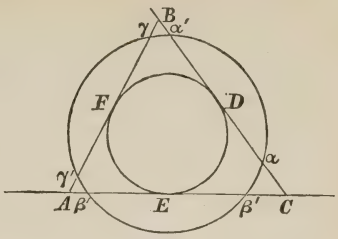

Fig. 7.

The chords $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$ are therefore equal to one another, and consequently equidistant from the centre of the circle.

It appears therefore that in this case the circle is concentric with the inscribed circle of the triangle ABC.

It can easily be seen that

$$\left.\begin{aligned} A\beta &= A\gamma' \\ B\gamma &= B\alpha' \\ C\alpha &= C\beta' \end{aligned}\right\}$$

and that

$\beta\gamma'$ is parallel to $\beta'\gamma$,

$\gamma\alpha'$ „ „ $\gamma'\alpha$,

$\alpha\beta'$ „ „ $\alpha'\beta$.

It will be seen that the point O in this case is the intersection of AD, BE, CF, where D, E, F are the points of contact of the inscribed circle of the triangle ABC.

In Case I.,

$$\left.\begin{aligned} \alpha = \alpha' &= \frac{\pi}{2} - \frac{A}{2} \\ \beta = \beta' &= \frac{\pi}{2} - \frac{B}{2} \\ \gamma = \gamma' &= \frac{\pi}{2} - \frac{C}{2} \end{aligned}\right\} \quad \text{.....(4).}$$

Therefore, by equation (11),

$$\frac{R}{r} = \frac{\cos\frac{B}{2}\sin(B+\theta)}{\sin A \sin B} + \frac{\cos\frac{C}{2}\cos\left(\frac{A}{2}+C-\theta\right)}{\sin A \sin C},$$

or, rearranging,

$$\frac{R}{r} = \frac{1}{4}\,\frac{\sin(\theta+\frac{1}{2}B)}{\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C} \quad \text{.....(27).}$$

Here, if $\theta = \frac{\pi}{2} - \frac{B}{2}$, we have the case of the inscribed circle, when, as is well known,

$$\frac{R}{r} = \frac{1}{4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.$$

It is evident that the inscribed circle is the minimum circle of Case I.

Case II., $\left.\begin{aligned} \alpha &= A \\ \beta &= B \\ \gamma &= C \end{aligned}\right\}$ .....(5)

(see Fig. 8).

Here, if $A\alpha$, $B\beta$, $C\gamma$ meet in a point, the point is the centre of the triangle ABC, and α, β, γ are the middle points of BC, CA, AB.

In this case we have the nine-point circle of the triangle ABC.

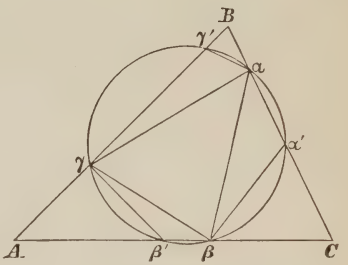

Fig. 8.

$$\angle A\gamma'\beta = \angle \gamma\gamma'\beta = \angle \gamma\alpha\beta = \alpha = A.$$

Therefore $\beta\gamma' = \beta A$, similarly $\beta\alpha' = \beta C$;
similarly also $\gamma\alpha' = \gamma B$, $\alpha\beta' = \alpha C$
$\gamma\beta' = \gamma A$, $\alpha\gamma' = \alpha B$ ..... (28).

It appears therefore that the perimeter of the hexagon $\alpha\beta'\gamma\,\alpha'\beta\gamma'$ is equal to the perimeter of the triangle ABC.

Again, from equation (11),

$$\frac{R}{r} = \frac{\sin(A+C-\theta)+\sin(A-C+\theta)}{\sin A}$$

$$= \frac{2\sin A\cos(C-\theta)}{\sin A} = 2\cos(C-\theta) = 2\sin\left(\frac{\pi}{2}+C-\theta\right) \ldots(29).$$

If $\theta = C$, we have the case of the nine-point circle, and $\frac{R}{r} = 2$.

It is evident that the nine-point circle is the minimum circle of Case II.

Case III.,

$$\left.\begin{aligned} \alpha' &= \gamma = B \\ \beta' &= \alpha = C \\ \gamma' &= \beta = A \end{aligned}\right\} \ldots(8).$$

Here $\beta'\gamma' = \alpha\beta$

(see Fig. 9),

therefore $\beta\gamma' = \alpha\beta'$,

therefore $\gamma'\alpha$ is parallel to $\beta\beta'$, *i.e.* to AC. Similarly $\alpha'\beta$ is parallel to AB and $\beta'\gamma$ to BC.

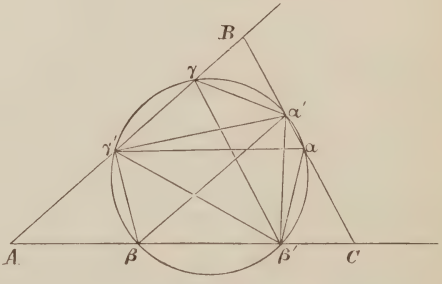

Fig. 9.

Again, in this case, because $C\alpha \,.\, C\alpha' = C\beta \,.\, C\beta'$,

$$\frac{C\alpha}{C\beta'} = \frac{C\beta}{C\alpha'} = \frac{CA}{CB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The perpendicular distances of $\beta'\gamma$ from BC, and of $\gamma'\alpha$ from CA, are $C\beta' \sin C$, and $C\alpha \sin C$ respectively; hence it follows that in this case the perpendicular distances of $\alpha'\beta$, $\beta'\gamma$, $\gamma'\alpha$ from the sides of the triangle ABC, AB, BC, CA respectively, are proportional to AB, BC, CA.

In the particular case when these three lines $\alpha'\beta$, $\beta'\gamma$, $\gamma'\alpha$ pass through a point, the circle becomes what has been called the "Triplicate Ratio" circle. (See *Quarterly Journal of Mathematics*, Vol. xix., p. 342.)

The point through which the three lines are drawn is the centre of similitude of the two triangles ABC, $A''B''C''$. (Fig. 2.)

In Case III. (Fig. 9), we see at once that the angles $B\alpha\gamma$, $C\beta\alpha$, $A\gamma\beta$ are equal, since we have

$$\alpha = C \quad \text{and} \quad \gamma = B.$$

In this case, therefore, the sides $\alpha\gamma$, $\beta\alpha$, $\gamma\beta$ of the triangle $\alpha\beta\gamma$ make the same angle θ with the sides BC, CA, AB respectively of the triangle ABC.

It will be observed that the angle θ is the inclination of the sides of the triangle $\alpha'\beta'\gamma'$ to those of ABC in the opposite direction.

Again, from equation (11),

$$\frac{R}{r} = \frac{\sin(A+C-\theta)}{\sin B} + \frac{\sin B \sin\theta}{\sin A \sin C} = \cos\theta + \sin\theta \left\{ \cot B + \frac{\sin B}{\sin A \sin C} \right\}$$

$$= \frac{\sin(\theta+\phi)}{\sin\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31),$$

if $$\cot\phi \equiv \cot B + \frac{\sin B}{\sin A \sin C} = \frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C}$$

$$= \cot A + \cot B + \cot C \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

If $\theta = \frac{\pi}{2}$, we have the case of the triangles $\alpha\beta\gamma$, $\alpha'\beta'\gamma'$, having their sides at right angles to those of ABC, when

$$\frac{R}{r} = \frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33).$$

(See *Trinity College Scholarship Examination Papers*, December, 1883.)

If $$\tan\theta = -\tan A \tan B \tan C,$$

then
$$\frac{R}{r} = \frac{\sin\theta}{\sin A \sin B \sin C}$$
$$= \surd\{\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C\} \quad \ldots\ldots\ldots(34).$$

This is the case where the points α, β, γ, α', β', γ' are the feet of the perpendiculars on the sides of ABC from D, E, F, the feet of the perpendiculars from A, B, C on the opposite sides. (See *Math. Messenger*, Vol. xi., p. 177.)

If $\quad \cot\theta = \cot A + \cot B + \cot C = \dfrac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C}$,

then $\dfrac{R}{r} = 2\cos\theta = \dfrac{2\,(1+\cos A \cos B \cos C)}{\surd\{(1+\cos A \cos B \cos C)^2 + \sin^2 A \sin^2 B \sin^2 C\}}$.

This is the case of the "Triplicate Ratio" circle.

The numerator of the last fraction
$$= \sin^2 A + \sin^2 B + \sin^2 C,$$
and the denominator
$$= \surd\{\sin^2 A \sin^2 B + \sin^2 B \sin^2 C + \sin^2 C \sin^2 A\},$$
therefore
$$\frac{R}{r} = \frac{a^2+b^2+c^2}{\surd(b^2c^2+c^2a^2+a^2b^2)} \quad \ldots\ldots\ldots(35),$$
which is the form given by Mr. Tucker in the *Quarterly Journal*, *loc. cit.*

Now, since
$$\frac{R}{r} = \frac{\sin(\theta+\phi)}{\sin\phi},$$
r is a minimum when $\theta+\phi = \dfrac{\pi}{2}$, in which case
$$\frac{R}{r} = \frac{1}{\sin\phi} = \surd\{1+(\cot A+\cot B+\cot C)^2\} \quad \ldots\ldots\ldots(36).$$

If $\beta\alpha'$ be parallel to AB,
$$\angle\,\beta'\alpha C = \angle\,\alpha'\beta C = \angle\,BAC = A,$$
and
$$\angle\,\alpha\beta' C = \angle\,\beta\alpha' C = B.$$

Hence we see that the triangle $\alpha\beta' C$ (and by symmetry each of the triangles $A\beta\gamma'$, $\alpha' B\gamma$) is similar to the triangle ABC, and
$$\gamma\alpha' = 2r\sin\theta = \alpha\beta' = \beta\gamma' = k \text{ suppose.}$$

Therefore
$$\left.\begin{aligned} \frac{B\gamma}{a} = \frac{B\alpha'}{c} = \frac{\gamma\alpha'}{b} \\ \frac{C\alpha}{b} = \frac{C\beta}{a} = \frac{\alpha\beta'}{c} \\ \frac{A\beta}{c} = \frac{A\gamma'}{b} = \frac{\beta\gamma'}{a} \end{aligned}\right\} \quad \ldots\ldots\ldots(37),$$

whence
$$\left.\begin{aligned} B\gamma = \frac{a}{b}k, \quad C\alpha = \frac{b}{c}k, \quad A\beta = \frac{c}{a}k \\ B\alpha' = \frac{c}{b}k, \quad C\beta' = \frac{a}{c}k, \quad A\gamma' = \frac{b}{a}k \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(38).$$

Therefore
$$A\beta \,.\, B\gamma \,.\, C\alpha = k^3 = A\gamma' \,.\, B\alpha' \,.\, C\beta' \ldots\ldots\ldots\ldots\ldots\ldots(39).$$

It is evident that, in Case III., the centre of the circle $\alpha\beta\gamma$ is the centre of the inscribed circle of the triangle formed by producing the three chords $\beta\gamma'$, $\gamma\alpha'$, $\alpha\beta'$, *i.e.* (see Fig. 2) of the triangle $A'B'C'$.

If we wish to find the point O for Case III., we may find the position where α, α' coincide.

This will happen when
$$B\alpha' + C\alpha = BC,$$

or
$$\frac{c}{b}k + \frac{b}{c}k = a\,;$$

therefore
$$k = \frac{abc}{b^2+c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(40).$$

If this point be called α'', we may write
$$B\alpha'' = \frac{ac^2}{b^2+c^2}, \qquad C\alpha'' = \frac{ab^2}{b^2+c^2}\,;$$

therefore
$$\frac{B\alpha''}{C\alpha''} = \frac{c^2}{b^2},$$

or α'' divides the side BC in the ratio $c^2 : b^2$, as was proved before.

On the Induction of Electric Currents in Cylindrical and Spherical Conductors. By Horace Lamb, M.A.

[*Communicated Jan.* 10*th*, 1884.]

The following calculations relate chiefly to some cases of induction of electric currents in a long circular cylinder by variation of the magnetic field in which it is placed, this field being supposed uniform, with the lines of force parallel to the axis of the cylinder. For example, we may suppose the cylinder placed in the interior of a long hollow coil, round which a variable electric current is made to circulate. In one instance I have added the solution of the corresponding problem for the sphere. The theory of the *free* transverse currents in a circular cylinder has been recently given by Lord Rayleigh.* The

* *British Association Reports* for 1882, p. 446.

first part of what follows is almost a transcript of his note, the method of which is simpler than that which I had originally adopted.

The axis of the cylinder being taken as axis of z, we have

$$\frac{dc}{dy} = 4\pi\mu u, \quad -\frac{dc}{dx} = 4\pi\mu v \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\rho u = -\frac{d\phi}{dx} - \frac{dF}{dt}, \quad \rho v = -\frac{d\phi}{dy} - \frac{dG}{dt} \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where c denotes the magnetic induction, which is assumed to be everywhere parallel to z, ρ the specific resistance of the material, μ its magnetic permeability, whilst u, v are the components of current, F, G those of electric momentum, parallel to x and y respectively.

Since
$$c = \frac{dG}{dx} - \frac{dF}{dy},$$
we obtain from (1) and (2)

$$\frac{dc}{dt} = \frac{\rho}{4\pi\mu}\left(\frac{d^2c}{dx^2} + \frac{d^2c}{dy^2}\right),$$

or, if c be assumed to vary as $e^{\lambda t}$, where λ is a constant,

$$\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + k^2\right) c = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

provided
$$k^2 = -\frac{4\pi\lambda\mu}{\rho} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Outside the cylinder we have, by (1), $c = \text{const.}, = I$, say.

If we put $x = r\cos\theta$, $y = r\sin\theta$, the solution of (3) is

$$c = \Sigma J_n(kr)\,(A_n\cos n\theta + B_n\sin n\theta) \ldots\ldots\ldots\ldots\ldots(5).$$

It appears from (1) that the lines of flow of the currents in the cylinder are the lines $c = \text{const.}$ At the surface ($r = R$) of the cylinder, the magnetic induction just inside must have μ times its value just outside, whence

$$\Sigma\,(A_n\cos n\theta + B_n\sin n\theta)\,J_n(kR) = \mu I \ldots\ldots\ldots\ldots\ldots(6).$$

In the case of *free currents*, we have $I = 0$, and thence

$$J_n(kR) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

When the admissible values of k have been found from (7), the corresponding values of λ are given by (4). For the slowest of the fundamental modes of decay, Lord Rayleigh finds the modulus $[-\lambda^{-1}]$ for a copper cylinder, one centimetre in radius, to be about $\frac{1}{800}$ of a second. In a cylinder of iron of the same size, the currents have much greater persistency, the inferior conductivity being much more than counterbalanced by the large value of μ. Assuming, roughly, $\mu = 400$, $\rho = 10000$ C. G. S., we have corresponding to the

lowest root of (7), in the case $n = 0$,

$$-\lambda^{-1} = \cdot 087 R^2.$$

Hence, in a cylinder one foot in diameter, the currents of the type in question would take about 20 seconds to fall to $1/e$ of their original strength.*

The foregoing theory is also applicable to the case of a *sector* of a cylinder bounded by any two radial planes, say the planes $\theta = 0$ and $\theta = \alpha$, viz., we then have

$$c = J_n(kr) \sin n\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

where the admissible values of n are determined by $\sin n\alpha = 0$. For instance, when $\alpha = 2\pi$ (*i.e.*, the continuity of the cylinder is interrupted by a thin non-conducting partition extending along a radial plane), we have $n = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, &c. The equation to determine k has the same form (7) as before.

Proceeding now to the case of induction by variation of the magnetic field, let us suppose, in the first instance, that this variation follows the simple harmonic law, say $\lambda = ip$, where $i = \sqrt{-1}$. The value of k is then prescribed by (4). When the cylinder is complete, we must have, in (6), $A_n = 0$, $B_n = 0$, except for $n = 0$, and so

$$c = \frac{J_0(kr)}{J_0(kR)} \mu I \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).\dagger$$

When the period of the variation in the field is long in comparison with the time of decay of free currents, we have $J_0(kr) = 1$, nearly, so that c is approximately constant and $= \mu I$ throughout the section of the cylinder. But, in the opposite extreme, when the oscillations in the intensity of the field are rapid in comparison with the decay of free currents, the induced currents extend only to a small depth beneath the surface of the cylinder, the inner strata (so to speak) being almost completely sheltered from electro-motive force by the outer ones. Writing $\qquad k^2 = (1-i)^2 q^2,$

where $$q^2 = \frac{2\pi\mu p}{\rho},$$

we have, when qr is large,‡

$$J_0(kr) = \text{const.} \times \frac{e^{qr} \cdot e^{i(qr - \frac{1}{4}\pi)}}{\sqrt{r}},$$

---

* As Poisson's law of magnetic induction, on which the equations (1) are based, is in the case of iron a very imperfect representation of the facts, these results are to be taken merely as rough illustrations.

† I here denotes the amplitude of the variations in the field. The time-factor is omitted for shortness.

‡ q being taken positively.

approximately, and thence

$$c = \mu I \cdot \sqrt{\frac{R}{r}}\, e^{q(r-R)+iq(r-R)} \quad \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

This indicates that the electrical disturbance in the cylinder consists of a series of waves propagated inwards with rapidly diminishing amplitude.

A similar treatment is applicable to the case of a cylinder divided longitudinally by thin non-conducting partitions along any number of radial planes. Thus, in the case of a cylinder divided by a single diametral plane ($y = 0$), we have, for values of θ between 0 and π,*

$$c = \Sigma B_n J_n(kr) \sin n\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots(11),$$

where the values of the constants B_n are to be found from

$$\Sigma B_n J_n(kR) \sin n\theta = \mu I = \frac{4\mu I}{\pi}\left(\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \&c.\right) \ldots(12),$$

viz., $B_n = 0$, when n is even,

and $$B_n = \frac{1}{n} \cdot \frac{4\mu I}{\pi J_n(kR)} \text{ when } n \text{ is odd} \quad \ldots\ldots\ldots(13),$$

The case where the variation in the field, instead of being simple harmonic, follows any arbitrary law, can be deduced from the foregoing, by Fourier's theorem. The final result would of course, in most cases, present a somewhat complicated appearance.

There is one case, however, a very interesting one from a physical point of view, which admits of much simpler treatment, viz., that of the currents induced by the sudden cessation of a previously constant field, *e.g.*, by the "break" of the current in a long solenoid surrounding the cylinder. If we integrate the equations (2) with respect to t over the time of the break (assumed to be infinitely short), we see that there is no instantaneous change in the values of F, G, and therefore of c, at any point of the conductor. At a time t after the break, we shall have

$$c = \Sigma C_p J_0(k_p r)\, e^{\lambda_p t},$$

where k_p is the p^{th} root of

$$J_0(kR) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7a),$$

and the corresponding value of λ_p is given by (4). The values of the constants C_p are to be determined by the condition that $c = \mu I$ when $t = 0$. Now, if k_p, k_q denote different roots of (7*a*), we have†

$$\int_0^R J_0(k_p r)\, J_0(k_q r)\, r\, dr = 0,$$

* For values of θ between π and 2π, the sign of the right-hand side must be changed.

† Rayleigh's *Sound*, §§ 203, 204.

$$\int_0^R \{J_0(k_p r)\}^2 r\,dr = \frac{R^2}{2} J_1^2(k_p R),$$

$$\int_0^R J_0(k_p r)\, r\,dr = \frac{R}{k_p} J_1(k_p R).$$

Hence we find $$C_p = \frac{2\mu I}{k_p R \,.\, J_1(k_p R)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

The total induction through the section of the cylinder, at time t, is

$$\int_0^R c \,.\, 2\pi r\,dr = 4\pi\mu I \,.\, \Sigma \frac{e^{\lambda_p t}}{k_p^2}.$$

or, if $\zeta_1, \zeta_2, \ldots$ denote the roots of $J_0(\zeta) = 0$, and

$$\tau = \frac{4\pi\mu R^2}{\rho},$$

$$\int_0^R c \,.\, 2\pi r\,dr = 4\pi R^2 \,.\, \mu I \,.\, \Sigma \frac{e^{-\zeta^2 t/\tau}}{\zeta^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Since $\Sigma\zeta^{-2} = \frac{1}{4}$, this gives $\pi R^2 \mu I$ for $t = 0$, which is right. The electromotive force round any circuit embracing the cylinder is found by differentiating (15) with respect to t, and changing the sign, viz., it is $$\rho I \,.\, \Sigma e^{-\zeta^2 t/\tau} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

The annexed Table gives the values of the total induction and of the electromotive force, calculated from (15) and (16) respectively, for various values of the ratio t/τ. The total induction is given in terms of its initial value; the electromotive force in terms of ρI.

t/τ	Total Induction.	Electromotive Force.
·00	1·0000	∞
·02	·7014	1·7332
·04	·5904	1·1430
·06	·5105	·8789
·08	·4470	·7195
·10	·3941	·6089
·20	·2178	·3168
·30	·1220	·1765
·40	·0684	·0989
·50	·0384	·0555
·60	·0215	·0311
·70	·0121	·0174
·80	·0068	·0098
·90	·0038	·0055
1·00	·0021	·0031

The strength of the current at a distance r from the axis is

$$-\frac{1}{4\pi\mu}\frac{dc}{dr}=\frac{I}{2\pi R}\Sigma\frac{J_1(k_p r)}{J_1(k_p R)}e^{-\zeta_p^2 t/\tau} \quad \ldots\ldots\ldots\ldots(17).$$

We may notice that immediately after the break there are no currents in the interior of the cylinder, but only at the surface, where we have a *current-sheet* in which the current per unit length is $I/4\pi$.

The *energy* per unit length of the cylinder is

$$\frac{1}{8\pi\mu}\int_0^R c^2 . 2\pi r\, dr=\frac{\mu I^2 R^2}{2}\Sigma\frac{e^{-2\zeta^2 t/\tau}}{\zeta^2} \quad \ldots\ldots\ldots\ldots(18).*$$

The solution of the corresponding problem for the sphere can be effected by means of formulæ given in a paper "On Electric Motions in a Spherical Conductor."† It is there shown that the free currents are of two distinct types. In the first type, with which alone we are here concerned, the components of currents are given by the formulæ

$$\left.\begin{aligned} u&=\frac{k^2}{4\pi}\psi_n(kr)\left(y\frac{d}{dz}-z\frac{d}{dy}\right)\chi_n\\ v&=\frac{k^2}{4\pi}\psi_n(kr)\left(z\frac{d}{dx}-x\frac{d}{dz}\right)\chi_n\\ w&=\frac{k^2}{4\pi}\psi_n(kr)\left(x\frac{d}{dy}-y\frac{d}{dx}\right)\chi_n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(19),$$

whilst, for the components of magnetic induction, we have, inside the sphere,

$$\left.\begin{aligned} a&=-(n+1)\psi_{n-1}(kr)\frac{d\chi_n}{dx}+n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dx}\chi_n r^{-2n-1}\\ b&=-(n+1)\psi_{n-1}(kr)\frac{d\chi_n}{dy}+n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dy}\chi_n r^{-2n-1}\\ c&=-(n+1)\psi_{n-1}(kr)\frac{d\chi_n}{dz}+n\frac{k^2r^{2n+3}}{2n+1\,.\,2n+3}\psi_{n+1}(kr)\frac{d}{dz}\chi_n r^{-2n-1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(20),$$

and outside

$$\left.\begin{aligned} a&=nR^{2n+1}\psi_n(kR)\frac{d}{dx}\chi_n r^{-2n-1}\\ b&=nR^{2n+1}\psi_n(kR)\frac{d}{dy}\chi_n r^{-2n-1}\\ c&=nR^{2n+1}\psi_n(kR)\frac{d}{dz}\chi_n r^{-2n-1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(21).$$

---

* The case where the intensity of the field suddenly rises from zero to I, being constant both before and after, is solved in exactly the same way.

† See *Phil. Trans.*, Part 2, 1883.

The origin is supposed taken at the centre of the sphere,

$$r^2 = x^2+y^2+z^2,$$

χ_n is a solid harmonic of positive integral degree n, $k^2 = -4\pi\lambda\mu/\rho$ as before, and

$$\psi_n(\zeta) = 1 - \frac{\zeta^2}{2\,.\,2n+3} + \frac{\zeta^4}{2\,.\,4\,.\,2n+3\,.\,2n+5}$$
$$= (-)^n\,.\,3\,.\,5\,\ldots\,2n+1\,.\,\left(\frac{d}{\zeta\,d\zeta}\right)^n \frac{\sin\zeta}{\zeta} \quad \ldots\ldots\ldots\ldots(22).$$

In particular,

$$\psi_0(\zeta) = \frac{\sin\zeta}{\zeta},$$

$$\psi_1(\zeta) = 3\left(\frac{\sin\zeta}{\zeta^3} - \frac{\cos\zeta}{\zeta^2}\right),$$

$$\psi_2(\zeta) = -15\left(\frac{1}{\zeta^3} - \frac{3}{\zeta^5}\right)\sin\zeta - 45\,\frac{\cos\zeta}{\zeta^4}.$$

In the case of $\mu = 1$, which I take first as simplest, the values of k are determined by $\psi_{n-1}(kR) = 0$ .....(23).

If k_p, k_q denote different roots of this, it may be shown, exactly as in the theory of Bessel's functions, that

$$\int_0^R \psi_n(k_p r)\,\psi_n(k_q r)\,r^{2n+2}\,dr = 0 \quad \ldots\ldots\ldots\ldots(24),$$

$$\int_0^R \{\psi_n(kR)\}^2\,r^{2n+2}\,dr = \frac{R^{2n+3}}{2}\{\psi_n(kR)\}^2 \quad \ldots\ldots\ldots(25),$$

$$\int_0^R \psi_n(kR)\,r^{2n+2}\,dr = \frac{2n+1}{k^2R^2}\,R^{2n+3}\,\psi_n(kR) \quad \ldots\ldots\ldots(26).$$

To determine the free currents due to the sudden cessation of a magnetic field of uniform intensity I, we have the condition that, when $t = 0$, $a = 0$, $b = 0$, $c = I$, throughout the sphere, the axis of z being taken parallel to the lines of force. Now the formulæ (20) make

$$xa + yb + zc = -n\,(n+1)\,\psi_n(kr)\,\chi_n \ldots\ldots\ldots(27).$$

Comparing, we see that $\chi_n = 0$, except for $n = 1$, in which case we must have $2\Sigma\psi_1(k_p r)\,\chi_1 = -Iz$ .....(28),

the summation embracing the various values of k_p determined from

$$\psi_0(k_p R) = \frac{\sin k_p R}{k_p R} = 0.$$

Putting $\chi_1 = C_p z$, multiplying both sides of (28) by $\psi_1(k_p r)\,r^4$, and integrating with respect to r between 0 and R, we find

$$C_p = -\frac{3I}{k_p^2 R^2\,.\,\psi_1(k_p R)} \ldots\ldots\ldots(29).$$

Substituting in (21), we find that the magnetic field outside the sphere is that due to a small magnet whose moment

$$= 3IR^3\Sigma\,\frac{e^{-\nu^2 t/\tau}}{\zeta^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30),$$

where $\tau = 4\pi R^2/\rho$, and the ζ are the various roots of $\psi_0(\zeta) = 0$, viz., $\zeta_p = p\pi$. Since $\Sigma\zeta^{-2} = \frac{1}{6}$, this is $= IR^3/2$ for $t = 0$.

Again, we find from (20), for $z = 0$,

$$c = -\Sigma\left\{2\psi_0(kr) - \frac{k^2r^2}{15}\psi_2(kr)\right\}C_p$$

$$= -3\Sigma\left\{\left(\frac{1}{kr} - \frac{1}{k^3r^3}\right)\sin kr + \frac{\cos kr}{k^2r^2}\right\}C_p.$$

Hence the total induction through the equatorial plane is

$$\int_0^R c\,.\,2\pi r\,dr = 6\pi\Sigma C_p\,\frac{\cos k_p R}{k_p^2}$$

$$= 6\pi IR^2\Sigma\,\frac{e^{-\nu^2 t/\tau}}{\zeta^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31).$$

The electromotive force in a wire coinciding with the equator of the sphere is found by performing $-d/dt$ on (31).

Immediately after the break there is no current except at the surface of the sphere, viz., we have a *current-sheet* of strength

$$3I/8\pi\,.\,\sin\theta,*$$

where θ is the co-latitude [viz., $z = r\cos\theta$]. At any subsequent time t the components of current in the sphere are

$$\left.\begin{aligned} u &= -\frac{3I}{4\pi R^2}\,.\,y\,.\,\Sigma\,\frac{\psi_1(k_p r)}{\psi_1(k_p R)}\,.\,e^{-\zeta_p^2 t/\tau} \\ v &= \frac{3I}{4\pi R^2}\,.\,x\,.\,\Sigma\,\frac{\psi_1(k_p r)}{\psi_1(k_p R)}\,.\,e^{-\zeta_p^2 t/\tau} \\ w &= 0 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

The electrokinetic energy is

$$\tfrac{1}{2}\iiint(Fu + Gv)\,dx\,dy\,dz.$$

Making $F = 4\pi/k^2\,.\,u$, $G = 4\pi/k^2\,.\,v$, $dx\,dy\,dz = 2\pi r\sin\theta\,.\,r\,d\theta\,.\,dr$, substituting the values of u, v from (32), and integrating with respect to θ from 0 to π, and with respect to r from 0 to R, we find, by (24) and (25),

$$\text{Energy} = \tfrac{3}{2}I^2R^3\Sigma\,\frac{e^{-2\nu^2 t/\tau}}{\zeta^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33).$$

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* See Maxwell's *Electricity*, §§ 660, 672. By the "strength" of the current sheet is meant the current per unit length across a line drawn on the surface perpendicular to the lines of flow.

When the substance of the sphere is susceptible of magnetization the work is more complicated, and it may be sufficient to state the chief results. The formulæ (19), (20), (21) still hold, but (23) is replaced by

$$\psi_{n-1}(kR) + \frac{n}{2n+1}(\mu-1)\,\psi_n(kR) = 0 \quad \ldots\ldots\ldots\ldots(34).$$

The conditions to be satisfied at the time $t = 0$ are that

$$a = 0, \quad b = 0, \quad c = \frac{3\mu}{\mu+2} I,$$

throughout the sphere. We then find, instead of (29),

$$C_p = -\frac{3\mu I}{\psi_1(\zeta_p)} \cdot \frac{1}{\zeta_p^2 + (\mu-1)(\mu+2)} \quad \ldots\ldots\ldots\ldots(35),$$

where ζ_p is the p^{th} root of

$$\psi_0(\zeta) + \frac{\mu-1}{3}\psi_1(\zeta) = 0 \quad \ldots\ldots\ldots\ldots(36).$$

At the time t after the break, the magnetic field outside the sphere is that due to a small magnet whose moment

$$= 3\mu I R^3 \Sigma \frac{e^{-\zeta^2 t/\tau}}{\zeta^2 + (\mu-1)(\mu+2)} \quad \ldots\ldots\ldots\ldots(37),$$

where $\tau = 4\pi\mu R^2/\rho$. The total induction through the equatorial plane is

$$6\pi\mu I R^2 \Sigma \frac{e^{-\zeta^2 t/\tau}}{\zeta^2 + (\mu-1)(\mu+2)} \quad \ldots\ldots\ldots\ldots(38).$$

It may be shown, by the method of Lord Rayleigh's "Sound," § 332, that

$$\Sigma \frac{1}{\zeta^2 + (\mu-1)(\mu+2)} = \frac{1}{2(\mu+2)},$$

so that (38) assumes for $t = 0$ the value $3\mu I/(\mu+2) \cdot \pi R^2$, which is right.

The formulæ for the components of current are obtained from (32) by introducing the factor

$$\frac{k_p^2 R^2}{k_p^2 R^2 + (\mu-1)(\mu+2)}$$

under the sign Σ of summation. The initial current-sheet has the same strength $3I/8\pi \cdot \sin\theta$ as before.

The electrokinetic energy

$$= \tfrac{3}{2}\mu I^2 R^3 \Sigma \frac{e^{-2\zeta^2 t/\tau}}{\zeta^2 + (\mu-1)(\mu+2)}.$$

To illustrate the physical significance of this "electro-kinetic energy," we may consider the following question :—

A spherical portion of matter susceptible of magnetization by induction is slowly brought up from a distance into a magnetic field of uniform intensity I, and then suddenly removed. During the approach of the sphere there is a gain of work $= \frac{1}{2}MI$ .....[a], where M is the magnetic moment of the sphere when in the field I. During the instantaneous withdrawal a current-sheet is developed in the surface of the sphere, of strength such as to maintain the magnetic force in the interior unaltered. Hence work is spent in overcoming the attraction of the field, both on the current-sheet and on the magnetism of the sphere. The first part of the work is the same as if the sphere were devoid of magnetic susceptibility. It may be found, either by putting $t = 0$ in (33), or by direct calculation from the values $3I/8\pi \,.\, \sin\theta$ and $IR/2 \,.\, \sin\theta$ of the strength of the sheet, and of the electric momentum in it, at the instant after the withdrawal. In either way we obtain

$$\frac{I^2R^3}{4} \text{ .....[}b\text{]}.$$

The second part of the work $= IM$ .....[c].

The electro-kinetic energy immediately after the withdrawal is found by putting $t = 0$ in (39), viz., it is

$$\frac{3}{4}\,\frac{\mu}{\mu+2}\,I^2R^3 \text{ .....[}d\text{]}.$$

As the currents in the sphere decay, this energy gradually takes the form of heat, according to Joule's law. Since

$$M = (\mu-1)\,I^2R^3/(\mu+2),$$

we verify that $[a]-[b]-[c]+[d] = 0$.

Conversely, if the sphere be first suddenly brought into the field and afterwards slowly withdrawn, there is spent in the first instance an amount of work $= \frac{I^2R^3}{4}$ .....[a'], in overcoming the repulsion of the field on the current-sheet developed. As the currents in the sphere decay, there is generated an amount of heat equivalent to $\frac{3}{4}\,\frac{\mu}{\mu+2}\,I^2R^3$ .....[b']. At the same time the sphere gradually acquires the magnetic moment M. During the subsequent slow withdrawal, an amount of work $= \frac{1}{2}MI$ .....[c'] is spent. These quantities satisfy the relation

$$[a']-[b']+[c'] = 0.$$

The expressions [a] and [c'] are obtained by the following principle:—

If a body susceptible of inductive magnetization and destitute of coercive power be placed in the field due to any system of permanent magnets and constant electric currents, the amount of work necessary to slowly withdraw it from the field is half what would be required if it were to retain its magnetization unaltered. This principle follows easily from the expression

$$\tfrac{1}{2}\iiint \rho_1 V\,dx\,dy\,dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots [40],$$

given by Thomson* and Helmholtz† for the energy of any field in which we have both permanent magnets and bodies obeying Poisson's law of magnetic induction. In this formula ρ_1 denotes the density of the imaginary magnetic matter by which the *permanent* magnets may be replaced, and V the total magnetic potential; and the integration may be supposed to extend over all space. Hence, if we have only one magnetizable body B,‡ and if V_2 be the potential due to it, V_1 that due to the permanent magnets, so that $V = V_1 + V_2$ initially, the amount of work spent in slowly withdrawing the body, being equal to the increment of the function (40), is $-\tfrac{1}{2}\iiint \rho_1 V_2\,dx\,dy\,dz \ldots\ldots\ldots\ldots\ldots (41)$.

But, if ρ_2 be the density of the distribution of imaginary matter equivalent to the induced magnetization of B in its initial position, the expression (41) is, by Green's Theorem,

$$= -\tfrac{1}{2}\iiint \rho_2 V_1\,dx\,dy\,dz \ldots\ldots\ldots\ldots\ldots\ldots\ldots (42).\S$$

To make this proof applicable when we have constant electric currents as well as (or instead of) permanent magnets, we must suppose the currents replaced by equivalent magnetic shells. These must be arranged so as not to intersect the initial position of B.

March 13th, 1884.

Professor HENRICI, F.R.S., President, and subsequently S. Roberts, Esq., F.R.S., Vice-President, in the Chair.

The Rev. A. C. E. Blomfield, M.A. Oxon., Messrs. J. Chevallier, B.A. Camb., E. H. Hayes, M.A. Camb., R. S. Heath, B.A. Camb., and Professor J. Larmor, M.A. Camb., were elected members.

The following communications were made:—

"On the Square of Euler's Series," Mr. J. W. L. Glaisher, F.R.S.

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* Mathematical and Physical Papers, t. i., p. 530.

† *Wied. Ann.* t. xiii., p. 391; *Collected Papers*, t. i., p. 805.

‡ Or a number of such bodies rigidly connected together.

§ Compare Maxwell's *Electricity*, t. ii., p. 261, (second edition), where the above principle is proved in the particular case where either the size of B, or its inductive susceptibility, is so small that the disturbance produced by B itself in the field may be neglected in calculating the forces on it.

"Further results from a Theory of Transformation of Elliptic Functions," Mr. J. Griffiths.

"Note concerning the Pellian Equation," Mr. S. Roberts, F.R.S.

"On the Closed Link Polygons belonging to a system of Co-planar Forces having a single Resultant," Professor M. J. M. Hill, M.A.

"On the Direct Application of the Principle of Least Action to Dynamical Analogues," Professor J. Larmor.

The following presents were received :—

Cabinet likenesses of Dr. Helmholtz (presented by Mr. R. F. Scott), and of Mr. A. B. Kempe, F.R.S.

"Educational Times" for March.

"Mathematics from the Educational Times," Vol. xl.

"Biographical Notice of the late Prof. Challis," by J. W. L. Glaisher, 8vo pamphlet; London, 1883.

"Report of the British Association Committee on Mathematical Tables," 8vo pamphlet.

"Calculation of the Hyperbolic Logarithm of π," by J. W. L. Glaisher, 8vo pamphlet.

"Tables of the Exponential Function," by J. W. L. Glaisher, 4to pamphlet.

"Bulletin Astronomique et Météorologique de l'Observatoire Impérial de Rio de Janeiro," No. 10, October 1883.

"Berichte über die Verhandlungen der königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig—Mathematisch-physische Classe," 1882, 8vo; Leipzig, 1883.

"Elektrische Untersuchungen" von W. G. Hankel, Siebzehnte Abhandlung (Leipzig, 1883) des xii. Bandes der Abhandlungen, No. ix., pp. 599 to 650.

"Bulletin des Sciences Mathématiques et Astronomiques," December 1883 and January 1884.

"Atti della R. Accademia dei Lincei," Vol. 8, Fasc. 2, 3.

"Beiblätter zu den Annalen der Physik und Chemie, &c.," B. 8, St. 2.

"Sitzungsberichte der Physikalisch-medecinischen Societät zu Erlangen," 15 Heft, November 1882—August 1883; Erlangen, 1883.

"Crelle," Bd. xcvi., H. 1.

"Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften zu Berlin," xxxviii. to liii.

On the Closed Link Polygons belonging to a System of Co-planar Forces having a Single Resultant. *By* M. J. M. Hill, M.A., Professor of Mathematics at the Mason Science College, Birmingham.

[*Read March 13th*, 1884.]

It is known that, if a system of co-planar forces be in equilibrium, both the vector polygon and the link polygon are closed.

If the system be equivalent to a couple, the vector polygon is closed, the link polygon is not closed, its two extreme sides being parallel but non-coincident.

The object of this note is to show that, if the forces have a single resultant, the link polygon may be closed.

The known construction for the vector and link polygons is as follows :—

Let P_1, P_2, ... P_n be n forces in a plane having a single resultant.

To draw the vector polygon, take any point 01 in the plane. Draw a line from 01 to another point 12 representing P_1 in magnitude and direction. Draw a line from 12 to another point 23 representing P_2 in magnitude and direction.

And so on. The line representing P_n will be drawn from the point $\overline{n-1}\,n$ to the point $n\,\overline{n+1}$.

The points 01, 12, ... $n\,\overline{n+1}$ are called the vertices of the vector polygon.

Now take any pole O, and join it to these vertices.

Also, taking any point f_0 in the plane of the forces, draw

$f_0 f_1$ parallel to $01O$ cutting P_1 in f_1,

$f_1 f_2$ parallel to $12O$ cutting P_2 in f_2,

and so on,

$f_{n-1} f_n$ parallel to $\overline{n-1}\,nO$ cutting P_n in f_n,

and $f_n f_{n+1}$ parallel to $n\,\overline{n+1}\,O$.

The figure $f_0 f_1 f_2 \ldots f_n f_{n+1}$ is called a link polygon.

To show how to draw a closed link polygon, corresponding to such a system of forces.

It is known that the resultant can always be determined in magnitude and position, whenever the vector polygon and any link polygon whose extreme sides are not parallel have been drawn.

This having been done, let a new link polygon be drawn by taking the pole O on the line joining 01 to $n\,\overline{n+1}$, and the point f_0 on the resultant, so that $f_0 f_1$ is parallel to $01O$, and therefore coincides with the resultant.

By means of this link polygon when completed, the system of forces can be shown to be equivalent to two forces in $f_0 f_1$ and $f_n f_{n+1}$ respectively. These two forces being equivalent to the resultant, and one of them coinciding in position with the resultant, the other must do so also.

Therefore the two extreme sides of the link polygon coincide with each other, *i.e.*, the link polygon is closed.

It is evident also that there are no closed link polygons, whose coincident extreme sides do not coincide with the resultant.

The propositions mentioned in this paper, and those closely connected with them, may be brought together thus—

I. For coplanar forces in equilibrium :
The vector polygon is closed.
All link polygons are closed.

II. For coplanar forces equivalent to a couple :
The vector polygon is closed.
No link polygon is closed, but the two extreme sides of each link polygon are parallel and non-coincident.

III. For coplanar forces equivalent to a single resultant :
The vector polygon is not closed.
The link polygon may be—
(1) Closed.
The pole O is then on the line joining 01 to $n\,\overline{n+1}$; and f_0 is on the resultant.
or (2) Not closed, the two extreme sides being parallel to the resultant and non-coincident.
The pole O is then on the line joining 01 to $n\,\overline{n+1}$; and f_0 is not on the resultant.
or (3) Not closed, the two extreme sides intersecting on the resultant.
The pole O is then not on the line joining 01 to $n\,\overline{n+1}$.

Further Results from a Theory of Transformation of Elliptic Functions. *By* John Griffiths, M.A.

[*Read March 13th*, 1884.]

The theory in question is based on the assumption that a transformation-equation is composed of two or more lower equations. For example, the cubic equation $y = \dfrac{x\{1+2a'-(1+a')^2x^2\}}{1-a^2x^2}$, where $a^2+a'^2=1$, is made up of a linear equation $y_1 = x$ and a quadric one $y_2 = \dfrac{1-(1+a')x^2}{1-a^2x^2}$, in such wise that $y = y_1y_2 + \sqrt{1-y_1^2}\cdot\sqrt{1-y_2^2}$; or, say,

$$y = \sin(\theta+A), \text{ where } \sin\theta = x \text{ and } \cos A = \frac{1-(1+a')x^2}{1-a^2x^2}.$$

Hence it appears that Jacobi's first transformation can be presented in the form

$$\operatorname{am}(Mu, \lambda) = \operatorname{am}(u, k) + \Sigma \cos^{-1}\left\{\frac{1-(1+a')\operatorname{sn}^2(u,k)}{1-a^2\operatorname{sn}^2(u,k)}\right\},$$

where the coefficients α, β, &c. are of the form $k \operatorname{sn} \frac{2sK}{n}$; s being an integer extending from 1 to $\frac{1}{2}(n-1)$, and n an odd integer.

The *second* transformation may be developed in a similar manner. For instance, if we start with the equation $\tan \frac{A+\theta}{2} = a'.\tan\theta$, we have

$$y = \sin A = \frac{x\{2a'-1+(1-a')^2 x^2\}}{1-a^2x^2},$$

where

$$a = k \operatorname{sn} \frac{4\,si\,K'}{3};\quad a^2+a'^2 = 1.$$

This transformation-equation is also composed of a linear and a quadric one; for it may be written as

$$y = \sin(-\theta+A),$$

where $\sin\theta = x$ and $\cos A = \frac{1-(1+a')\,x^2}{1-a^2x^2}$.

Hence there is no difficulty in seeing that the second transformation may be condensed into the formula

$$(-)^{\frac{n-1}{2}} \operatorname{am}(M_1 u, \lambda_1) = \operatorname{am}(u,k) + \Sigma(-)^s \cos^{-1}\left\{\frac{1-(1+a'^2)\operatorname{sn}^2(u,k)}{1-a^2\operatorname{sn}^2(u,k)}\right\},$$

where α, &c. $\equiv k \operatorname{sn} \frac{2\,si\,K'}{n}$.

If each of the above addition-equations be differentiated with respect to u, we have, evidently,

$$M \operatorname{dn}(Mu,\lambda) = \operatorname{dn}(u,k)\left\{1+2\Sigma\frac{a'}{1-a^2\operatorname{sn}^2(u,k)}\right\}$$

and

$$(-)^{\frac{n-1}{2}} M_1 \operatorname{dn}(M_1 u, \lambda_1) = \operatorname{dn}(u,k)\left\{1+2\Sigma\frac{(-)^s a'}{1-a^2\operatorname{sn}^2(u,k)}\right\}.$$

Again, if we differentiate with respect to the modulus k, there results in each case a relation between the functions $E(u,\lambda)$, $E(u,\lambda_1)$ $E(u,k)$, and u. The result for the first transformation has been already given by me.

The first and second transformations are thus disposed of, and it only remains to consider those which correspond to the other roots of the modular equation, and also to show the connexion between these transformations taken all together and the theory of multiplication.

With regard to this I have arrived at the conclusion, that for a prime number n there must be the following theorem; viz.,

$$(-)^{\frac{n-1}{2}} \operatorname{am}(Mu,\lambda) + \operatorname{am}(M_1 u,\lambda_1) + \operatorname{am}(M_2 u,\lambda_2) + \ldots = \operatorname{am}(nu,k) + (-)^{\frac{n-1}{2}} n \operatorname{am}(u,k);$$

where M, λ; M_1, λ_1; M_2, λ_2; &c., refer to the first, second, third trans-

formations, and so on for all the moduli corresponding to a given equation between λ and k.

For instance, in the cubic transformation there are four roots λ, λ_1, λ_2, λ_3, corresponding to a given value of the modulus k, and hence

$$-\mathrm{am}\,(Mu,\lambda)+\mathrm{am}\,(M_1u,\lambda_1)+\mathrm{am}\,(M_2u,\lambda_2)+\mathrm{am}\,(M_3u,\lambda_3)$$
$$=\mathrm{am}\,(3u,k)-3\mathrm{am}\,(u,k).$$

Here we may assume $3\Lambda = M\,.\,K$, $\Lambda_1 = M_1\,.\,K$, $\Lambda_2 = M_2\,.\,K$, and $\Lambda_3 = M_3\,.\,K$, so that

$$-\mathrm{am}\,(3\Lambda,\lambda)+\mathrm{am}\,(\Lambda_1,\lambda_1)+\ldots=\mathrm{am}\,(3K)-3\,\mathrm{am}\,K=0,$$

i.e.,
$$-3\,.\,\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}=0.$$

The equation

$$(-)^{\frac{1}{2}(n-1)}\mathrm{am}\,(Mu,\lambda)+\Sigma\,\mathrm{am}\,(M_1u,\lambda_1)=\mathrm{am}\,(nu,k)+(-)^{\frac{1}{2}(n-1)}n\,\mathrm{am}\,(u,k)$$

leads to a very interesting property of the modular equation, corresponding to a prime number n, which I have verified for $n=3$, 5 and 7.

It is this: let $k'=\tan^2 u$ and $\lambda'=\tan^2 v$, then the equation (of the $\overline{n+1}^{\text{th}}$ degree) connecting $\tan v$ and $\tan u$ is such that the sum of its roots
$$\Sigma v=(n+1)\left\{1+(-)^{\frac{1}{2}(n+1)}\right\}\frac{\pi}{4}+(-)^{\frac{1}{2}(n-1)}(n+1)\,u.$$

For example, let $n=3$, then the modular equation is

$$\sqrt{k\lambda}+\sqrt{k'\lambda'}=1,$$

or $\tan^4 v-4\tan^3 u\tan^3 v+6\tan^2 u\tan^2 v-4\tan u\tan v+\tan^4 u=0$,

so that
$$\tan(\Sigma v)=\frac{4\tan^3 u-4\tan u}{1-6\tan^2 u+\tan^4 u}=-\tan 4u.$$

i.e.,
$$\Sigma v=4\,.\,2\,.\,\frac{\pi}{4}-4u=2\pi-4u;$$

it being assumed that when $u=\frac{\pi}{4}$ we have

$$v=v_1=v_2=v_3=\frac{\pi}{4}.$$

If $n=5$, the formula gives $\Sigma v=6\times 0+6u=6u$; the modular equation in this case is

$$\tan^6 v-(16\tan^5 u-10\tan u)\tan^5 v+15\tan^2 u\tan^4 v-20\tan^3 u\tan^3 v$$
$$+15\tan^4 u\tan^2 v-(16\tan^5 u-10\tan^5 u)\tan v+\tan^6 u=0.$$

Hence
$$\tan(\Sigma v)=\frac{16\tan^5 u-10\tan u-20\tan^3 u+16\tan u-10\tan^5 u}{1-15\tan^2 u+15\tan^4 u-\tan^6 u}$$
$$=\frac{6\tan u-20\tan^3 u+6\tan^5 u}{1-15\tan^2 u+15\tan^4 u-\tan^6 u}=\tan 6u;$$

therefore $\Sigma v=6u$.

If $n=7$:
$$\Sigma v=8\times 2\times\frac{\pi}{4}-8u=4\pi-8u;$$

$$v=v_1=v_2=\ldots=v_7=\frac{\pi}{4},\quad\text{when}\quad u=\frac{\pi}{4}.$$

But the modular equation is

$$\tan^8 v-(64\tan^7 u-56\tan^3 u)\tan^7 v+(140\tan^2 u-112\tan^6 u)\tan^6 v$$
$$-(112\tan^5 u-56\tan u)\tan^5 v+70\tan^4 u\tan^4 v$$
$$-(112\tan^3 u-56\tan^7 u)\tan^3 v+(140\tan^6 u-112\tan^2 u)\tan^2 v$$
$$-(64\tan u-56\tan^5 u)\tan v+\tan^8 u=0.$$

If $u=\frac{\pi}{4}$, then $(\tan v-1)^8=0$, and it can be verified without difficulty that $\tan(\Sigma v)=-\tan 8u$, *i.e.*, that $\Sigma v=4\pi-8u$.

These results flow immediately from the equation given above; for we have, by a first differentiation with respect to u,

$$(-)^{\frac{1}{2}(n-1)}M\,\mathrm{dn}\,(Mu,\lambda)+\Sigma M_1\,\mathrm{dn}\,(M_1u,\lambda_1)=n\,\mathrm{dn}\,(nu,k)$$
$$+(-)^{\frac{1}{2}(n-1)}n\,\mathrm{dn}\,(u,k),$$

and so

$$(-)^{\frac{1}{2}(n-1)}M^2\lambda^2\,\mathrm{sn}\,(Mu,\lambda)\,\mathrm{cn}\,(Mu,\lambda)+\Sigma M_1^2\lambda_1\,\mathrm{sn}\,(M_1u,\lambda_1)\,\mathrm{cn}\,(M_1u,\lambda_1)$$
$$=nk^2\{n\,\mathrm{sn}\,(nu,k)\,\mathrm{cn}\,(nu,k)+(-)^{\frac{1}{2}(n-1)}\,\mathrm{sn}\,(u,k)\,\mathrm{cn}\,(u,k)\}.$$

Let $u=\frac{K}{2}$; then $M\frac{K}{2}=n\frac{\Lambda}{2}$, $M_1\frac{K}{2}=\frac{\Lambda_1}{2}$, &c.,

$$\mathrm{sn}\left(n\frac{\Lambda}{2},\lambda\right)\mathrm{cn}\left(n\frac{\Lambda}{2},\lambda\right)=(-)^{\frac{1}{2}(n-1)}\cdot\frac{\sqrt{\lambda'}}{1+\lambda'};$$

$M^2=\frac{nk\,k'^2}{\lambda\lambda'^2}\frac{d\lambda}{dk}$, for every root of the modular equation—so that it is easily seen we have

$$\Sigma\frac{d\cdot\sqrt{\lambda'}}{1+\lambda'}=(-)^{\frac{1}{2}(n-1)}(n+1)\frac{d\cdot\sqrt{k'}}{1+k'},$$

or $\Sigma\,dv=(-)^{\frac{1}{2}(n-1)}(n+1)\,du$, if $\sqrt{k'}=\tan u$, $\sqrt{\lambda}=\tan v$.

Hence $\Sigma v=(n+1)\{1+(-)^{\frac{1}{2}(n+1)}\}\frac{\pi}{4}+(-)^{\frac{1}{2}(n-1)}(n+1)\,u$,

where $v=v_1=v_2=\ldots=\frac{\pi}{4}$ when $u=\frac{\pi}{4}$.

It thus appears that there is an addition-theorem for moduli as well as for amplitudes.

As an illustration of the manner in which multiplication presents itself in the present theory, I consider here the problem of triplication, and so verify the formula given above, viz.,

$$(-)^{\frac{1}{2}(n-1)}am\,(Mu,\lambda)+\Sigma am\,(M_1u,\lambda_1)=am\,(nu,k)+(-)^{\frac{1}{2}(n-1)}n\,am\,(u,k).$$

I start with the expression

$$y=\sin(-A+B+C+D),$$

where

$$\cos A=\frac{1-(1+a'^2)\,x^2}{1-a^2x^2},$$

and

$$a=k\,\mathrm{sn}\frac{4K}{3},\quad a'=\mathrm{dn}\frac{4K}{3},$$

$$\sin B = \frac{x\{2\beta' - 1 + (1-\beta')^2 x^2\}}{1-\beta^2 x^2}, \quad \beta = k\,\mathrm{sn}\,\frac{4iK'}{3}, \quad \beta' = \mathrm{dn}\,\frac{4iK'}{3}$$

$$\cos C = \frac{1-(1+\gamma')\,x^2}{1-\gamma^2 x^2}, \quad \gamma = k\,\mathrm{sn}\,\frac{4K+4iK'}{3}, \quad \gamma' = \mathrm{dn}\,\ldots,$$

$$\cos D = \frac{1-(1+\delta')\,x^2}{1-\delta^2 x^2}, \quad \delta = k\,\mathrm{sn}\,\frac{-4K+4iK'}{3}, \quad \delta' = \mathrm{dn}\,\ldots;$$

and thus, in accordance with the theory, there results the differential expression

$$\frac{dy}{\sqrt{1-y^2}\,.\,1-\lambda^2 y^2} = N\frac{dx}{\sqrt{1-x^2}\,.\,1-k^2x^2}.$$

Hence, if it can be shown that $\lambda = k$ and $N = 3$, it is clear that this will be a triplication.

Now, if $x=1$ or $x = \sin\theta = 1$, and $y = \sin\phi$, it is seen at once that corresponding to the value $\theta = \frac{\pi}{2}$, we have

$$A = C = D = 2\,\frac{\pi}{2} \quad \text{and} \quad B = \frac{\pi}{2},$$

so that $$\phi = -\,2\,\frac{\pi}{2} + \frac{\pi}{2} + 2\,\frac{\pi}{2} + 2\,\frac{\pi}{2} = 3\,\frac{\pi}{2},$$

and thus $3\Lambda = N\,.\,K$. Again, there is no difficulty in finding that

$$N = -\,2\alpha' + 2\beta' - 1 + 2\gamma' + 2\delta',$$

and all that remains is to verify the value $N = 3$. To effect this, I observe that the roots of the equation

$$z^4 - 4z^3 + 6k'^2z^2 - 4k'^4z + k'^4 = 0$$

are α'^2, β'^2, γ'^2, and δ'^2. Hence it has to be shown that the roots z_1, z_2, z_3, z_4 can be arranged so that

$$-\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3} + \sqrt{z_4} = 2.$$

Assuming that this is so, we have

$$z_2 + z_3 + z_4 + 2\sqrt{z_3z_4} + 2\sqrt{z_4z_2} + 2\sqrt{z_2z_3} = 4 + 4\sqrt{z_1} + z_1,$$

or $$4 - z_1 + 2\sqrt{z_3z_4} + 2\sqrt{z_4z_2} + 2\sqrt{z_2z_3} = 4 + 4\sqrt{z_1} + z_1,$$

$$\sqrt{z_3z_4} + \sqrt{z_4z_2} + \sqrt{z_2z_3} = 2\sqrt{z_1} + z_1,$$

$$z_3z_4 + z_4z_2 + z_2z_3 + 2\sqrt{z_2z_3z_4}\,(\sqrt{z_2} + \sqrt{z_3} + \sqrt{z_4}) = 4z_1 + 4z_1^{\frac{3}{2}} + z_1^2,$$

$$6k'^2 - z_1\,(z_2 + z_3 + z_4) + 2\,\frac{k'^2}{\sqrt{z_1}}\,(2 + \sqrt{z_1}) = 4z_1 + 4z_1^{\frac{3}{2}} + z_1^2,$$

$$6k'^2 - z_1\,(4 - z_1) + 2k'^2 + 4k'^2z_1^{-\frac{1}{2}} = 4z_1 + 4z_1^{\frac{3}{2}} + z_1^2,$$

$$8k'^2-8z_1 = 4z_1^{\frac{3}{2}}-4k'^2z_1^{-\frac{1}{2}},$$

$$(2k'^2-2z_1)^2 = (z_1^{\frac{3}{2}}-k'^2z_1^{-\frac{1}{2}})^2, \quad i.e., \; z_1^4-4z_1^3+6k'^2z_1^2-4k'^4z_1+k'^4 = 0.$$

The verification is thus completed, and we have $\Lambda = K$, which is satisfied by $\lambda = k$.

Again, to return to the different cubic transformations, we have, for the first,

$$\operatorname{am}(Mu, \lambda) = \operatorname{am}(u, k)+\cos^{-1}\left\{\frac{1-(1+a'^2)\operatorname{sn}^2(u, k)}{1-a^2\operatorname{sn}^2(u, k)}\right\},$$

where $$a = k\operatorname{sn}\frac{4K}{3},$$

or $$\operatorname{am}(Mu, \lambda) = \operatorname{am}(u, k)+A.$$

For the second,

$$\operatorname{am}(M_1u, \lambda_1) = -\operatorname{am}(u, k)+\cos^{-1}\left\{\frac{1-(1+\beta'^2)\operatorname{sn}^2(u, k)}{1-\beta^2\operatorname{sn}^2(u, k)}\right\},$$

where $$\beta = k\operatorname{sn}\frac{4iK'}{3}, \quad \text{or} \quad \operatorname{am}(M_1u, \lambda_1) = B.$$

Similarly for the third transformation,

$$\operatorname{am}(M_2u, \lambda_2) = -\operatorname{am}(u, k)+\cos^{-1}\left\{\frac{1-(1+\gamma'^2)\operatorname{sn}^2(u, k)}{1-\gamma^2\operatorname{sn}^2(u, k)}\right\},$$

where $$\gamma = k\operatorname{sn}\frac{4K+4iK'}{3},$$

or $$\operatorname{am}(M_2u, \lambda_2) = -\operatorname{am}(u, k)+C.$$

For the fourth, we have

$$\operatorname{am}(M_3u, \lambda_3) = -\operatorname{am}(u, k)+D.$$

Hence, since it has been shown that

$$\operatorname{am}(3u, k) = -A+B+C+D,$$

there results the formula

$$\begin{aligned}\operatorname{am}(3u, k) = &-\operatorname{am}(Mu, \lambda)+\operatorname{am}(u, k)+\operatorname{am}(M_1u, \lambda)\\ &+\operatorname{am}(M_2u, \lambda_2)+\operatorname{am}(u, k)\\ &+\operatorname{am}(M_3u, \lambda_3)+\operatorname{am}(u, k),\end{aligned}$$

or $$\begin{aligned}\operatorname{am}(3u, k)-3\operatorname{am}(u, k) = &-\operatorname{am}(Mu, \lambda)+\operatorname{am}(M_1u, \lambda_1)\\ &+\operatorname{am}(M_2u, \lambda_2)\\ &+\operatorname{am}(M_3u, \lambda_3),\end{aligned}$$

where λ, λ_1, λ_2, λ_3 are the four roots of the modular equation in the cubic transformation.

The formula

$$\operatorname{am}(nu, k)+(-)^{\frac{1}{2}(n-1)}n\,.\operatorname{am}(u, k) = (-)^{\frac{1}{2}(n-1)}\operatorname{am}(Mu, \lambda)+\Sigma\operatorname{am}(M_1u, \lambda_1)$$

(where M, λ; M_1, λ_1, &c. refer to the first, second, &c. transformation) has thus been proved to be true for $n = 3$; and, as a consequence of it, I have shown that, in the case of a prime number, there is an addition-theorem for the various moduli. The latter has also been verified for $n = 3$, 5, and 7.

As another result of the formula, we may notice $\Sigma M = 0$ or $2n$.

I may also observe that the triplication formula

$$y = \sin(-A+B+C+D)$$

throws some light on the modular equation for the nonic transformation.

This, according to the present theory, may be considered as composed of a cubic and three quadric transformations, and so is, in fact, of the form $y = \sin(A+B+C+D)$.

We should thus expect to find that $\lambda = k$ is a root of the modular equation. This is what actually happens, as may be seen from a memoir by Prof. Cayley "On the Transformation of Elliptic Functions," *Phil. Trans.*, t. 164 (1874). The above result is in complete accord with Jacobi's theorem as to the transformations corresponding to a square number.

On the Immediate Application of the Principle of Least Action to Dynamics of a Particle, Catenaries, and other related Problems.

By Professor J. Larmor.

[*Read March 13th*, 1884.]

1. It is matter of constant observation that different departments of Mathematical Physics are closely connected together, so that the solution of a question in one branch of the subject admits of being transferred into another branch, and serving as the solution of a corresponding problem there.

To facilitate the general discussion of these relations between different problems, it is desirable to express each in its simplest mathematical terms, so that it shall involve as little reference as possible to the detailed theory of the special subject. Now nearly every such question can be expressed as a problem of maximum or minimum; and this mode of statement probably conveys a clearer and more compact representation of the question, and an easier grasp of its mathematical relations as a whole, than any other. It at once places the question beyond the accident of the particular coordinate system employed. We should therefore expect that it would be a direct and easy method of arriving at dynamical analogies.

The development of this method is the object of the following paper. Many of the examples given are more or less known, but there does not seem to have been any systematic attempt to group them together as illustrations of a single general principle.

2. The complete mathematical statement of every question in Dynamics is involved in either of the forms of the Principle of Least Action, when once the expressions for the potential and kinetic energies of the system are known. The principle admits of easy and elementary proof, and therefore forms a suitable foundation from which to construct dynamical theory. See, for example, Thomson and Tait's *Nat. Phil.*, Vol. i., Art. 327.

The prominent place that this principle directly holds in general dynamical theory is illustrated by the attempts that have been made to place the Second Law of Thermodynamics on a dynamical basis. The investigations of Clausius, Boltzmann, and Szily have shown that the Principle of Least Action in Dynamics is in close relationship with that law. And this agrees with what we might have expected from the above; for the law must have some dynamical interpretation, and the only dynamical relations hitherto discovered that are absolutely general and independent of coordinate systems of representation are the Principle of Energy, of which the First Law of Thermodynamics is a case, and the Principle of Least Action.

Hamilton pointed out the importance and began the development of the general method of reducing the solution of every dynamical question to the consideration of a single function (*Phil. Trans.*, 1834–1835); and it has since been widely extended and carried on by others (*cf.* Cayley's *Report*, *Brit. Assoc.*, 1857). What we now wish to point out is the ease and simplicity with which the mere statement of the principle, without any consequent analytical work, reveals the existence and nature of many analogies between different problems, whose relationship has been hitherto treated in a more or less isolated manner.

When we wish to compare the motion of one system with the equilibrium of another, we have the variational energy condition of equilibrium at hand, which may, if we please, be itself considered to be a case of the general principle.

3. In what follows we shall develope the relations between different problems relating to the motion of a particle, and the analogies between the motion of a particle and the forms of systems of rays and of catenaries that flow immediately from the minimum principle. The application of the method of inversion and of other methods of transformation flows naturally from it, and we are guided immediately to a large number of generalizations of solvable cases of motion

that have formed the subjects of separate papers by Liouville and others.

It may be observed that the method for brachistochrones, given, from Woodhouse, in the Appendix to Tait and Steele's *Dynamics*, affords also a very simple and elementary geometrical proof of the Principle of Least Action for a particle. For, drawing the system of equipotential surfaces of the field of force, the velocity of the particle between any two consecutive surfaces is invariable, as the energy of the motion is given. To determine that path which makes $\int v\,ds$ least, we take three consecutive points on the path, and find the position that the intermediate one must have on the equipotential through it, in order to make the variation of the integral zero. We shall be led to the ordinary law of centrifugal force in the orbit; and it follows that the actual orbit is the one possessing the stationary property.

In another paper the same principle will be applied to the discussion of problems connected with general solid and fluid systems, cyclic and acyclic, and of certain close analogies with questions relating to elastic wires and electric currents.*

Application of the Method of Inversion.

4. When a particle moves in a given field of force, the form of its orbit is determined by the condition that $\int v\,ds$ satisfies the maximum-minimum condition, in which v is given by the equation of energy

$$\tfrac{1}{2}v^2 + V = E, \text{ a constant,}$$

where V is the potential of the field of force.

Now, if we invert the orbit with respect to any origin, with radius of inversion k, $\int v\,ds$ is equivalent to $k^2 \int \frac{v}{r'^2}\,ds'$, where dotted letters refer to the inverse orbit; since, by Geometry,

$$ds : ds' = r : r' = k^2 : r'^2.$$

This new integral therefore satisfies the maximum-minimum condition, when taken along the inverse curve; and therefore this curve is the orbit of a particle in a field of force which makes

$$v' \propto v r'^{-2},$$

and therefore

$$\tfrac{1}{2}v'^2 \propto (E - V)\, r'^{-4},$$

in which V is now to be expressed in terms of the dotted letters.

The value of V', the potential of the new field of force, is therefore given by

$$AV' + (E - V)\, r'^{-4} = \text{constant},$$

or, what is the same,

$$r'^2 (E' - V') \propto r^2 (E - V).$$

* The original paper has been divided into two, and some other changes have been made, at the suggestion of the referee.

The velocity of projection in the new orbit is determined by the above value of v'.

We notice that, if the original law of force only contain powers of the coordinates higher than the inverse fifth, this velocity-condition will in general be that the particle be projected with the velocity from infinity. (See ex. 2, § 5.)

The ratio of the velocities at corresponding points of the related orbits is clearly inversely proportional to that of the radii vectores.

5. The following are examples :—

(1) With no forces acting, a particle constrained to remain on a fixed surface describes a geodesic with constant velocity; the curve corresponding to this geodesic on the inverse surface satisfies the relation $\delta \int r^{-2}\, ds = 0$, and is therefore a free path for a particle constrained to remain on the surface, under the action of a force varying as the inverse fifth power of the distance from the centre of inversion, and projected with the velocity from infinity.

On a sphere, the geodesics are great circles. Therefore particles projected on the surface of a sphere under the action of a force varying inversely as the fifth power of the distance from any fixed centre, and with the velocity due to approach from infinity, will describe circles on the sphere, which are its curves of intersection with the system of spheres passing through the centre of force and its inverse point. In particular, if the original sphere becomes a plane, we get Newton's result for free motion in a circle with a centre of force on its circumference.

On a cylinder of any form of cross-section, the geodesics make equal angles with the generators. Therefore on the ring of vanishing aperture, which is the inverse of such a cylinder, the paths of particles projected under the same conditions as before (the centre of force being now at the aperture) will be curves, each of which cuts all the meridians of the surface at the same angle. This result holds for any ring with circular meridians all passing through the same point; in particular, for a common anchor ring of vanishing aperture.

Again, lines of curvature invert into lines of curvature, and umbilici into umbilici; for the directions of the lines of curvature at a point are determined by the nature of the contact with spheres touching at the point, and spheres invert into spheres. Therefore we arrive at the result that, on any inverse of an ellipsoid, and in particular on Fresnel's Surface of Elasticity (in which case the centre of the surface is the centre of force), all particles projected from one umbilicus under the law of the inverse fifth power and with the velocity from infinity, will, when reflected by a line of curvature, pass through the other umbilicus, will then be again reflected through the first, and

so on continually. If the orbit pass through the two umbilici without reflection, it will have the form of a curve winding round and round the surface, and passing through them once in each revolution; if it do not pass through either, it will be such a curve touching the two branches of a line of curvature, one surrounding each umbilicus, once in each revolution. All this, of course, results from the well-known properties of geodesics on an ellipsoid.

Further, the orbits of particles moving on a surface under this law of force with any velocity of projection whatever, become on inversion the orbits described under similar conditions on the inverse surface; while, with the particular velocity discussed above, they become geodesics.

(2) For free motion in a plane the application is easy. Thus, if for the original orbit, the force (repulsive) $= \mu r^n$, we have

$$\tfrac{1}{2}v'^2 = \left\{ E + \frac{\mu}{n+1}\left(\frac{k^2}{r'}\right)^{n+1} \right\} r'^{-4},$$

the centre of inversion being the centre of force; and therefore in the inverted orbit the force $= \mu_1 r'^{-5} + \mu_2 r'^{-n-6}$, where the values of $\mu_1 \mu_2$ are obvious. If $n > -5$, the velocity will be that from infinity; while, if the velocity in the original orbit was that from the origin when $n > -1$, or to infinity when $n < -1$, we have the further simplification $\mu_1 = 0$.

We may also invert with respect to a point not in the plane of the orbit, and so obtain the fields of force in which curves drawn on a sphere are free orbits with proper velocity of projection. There is no need to supply examples.

6. *Transformation by Conjugate Functions.*

If a plane area is transformed by writing ξ, η for x, y, each element of length near a point is altered in the ratio λ to 1, where

$$\lambda^2 \equiv \frac{d(\xi\eta)}{d(xy)} \equiv \left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2 \equiv \left(\frac{d\eta}{dx}\right)^2 + \left(\frac{d\eta}{dy}\right)^2,$$

for in the original case ξ, η were the coordinates, but they are now replaced by x, y.

Hence the relation $\qquad \delta \int v\, ds = 0,$

becomes $\qquad \delta \int \lambda v\, d\sigma = 0,$

where $d\sigma$ is the corresponding element of arc of the transformed curve. The transformed curve is therefore the orbit, with proper velocity, in a field of force given by

$$V' \propto (E - V)\lambda^2.$$

For instance, if the orbit be the straight line

$$x + ay = b,$$

we have $V = \text{constant}$,

and therefore the curve $\xi + a\eta = b$

can be described freely in a field $V' \propto \lambda^2$.

On making $\xi = n \log r$, $\eta = n\theta$, we find $\lambda^2 = n^2 r^{-2}$, and therefore the spiral $r = Ae^{-a\theta}$ is the orbit under the law of the inverse cube, with velocity from infinity, as Newton proved.

7. We have here also a ready method of inventing fields of forces, in which the motion can be integrated. For example, motion in a field whose potential is given by

$$V = f(x) + F(y)$$

can obviously be reduced to quadratures. The transformed curves are free orbits in a field whose potential is

$$V' = \lambda^2 \{f(\xi) + F(\eta) - E\},$$

with proper velocity, E being the total energy of the original orbit. But further, being given this law of potential, without *any* velocity-condition, and transforming back again, we arrive at the law

$$V = f(x) + F(y) - E + C\lambda^{-2},$$

where C depends on the velocity; so that, if λ^{-2} is of the form $\phi(x) + \psi(y)$, we can integrate without any restrictions as to initial velocity.

Liouville arrived at these results from a discussion of the equations of motion.

(1) He applies them to the case of motion under two centres of force, by using elliptic coordinates with these centres as foci; thus, if μ, ν represent the semi-axes of the confocals through a point, and $2b$ the distance between the foci, we have

$$r = \mu + \nu, \quad r' = \mu - \nu,$$
$$ds^2 = (\mu^2 - \nu^2)\{(\mu^2 - b^2)^{-1} d\mu^2 + (b^2 - \nu^2)^{-1} d\nu^2\},$$

so that λ^{-2} is here equal to $\mu^2 - \nu^2$, and we can reduce to quadratures motion under any law

$$V = (\mu^2 - \nu^2)^{-1}\{f(\mu) + F(\nu)\} = (rr')^{-1}\{f(r+r') + F(r-r')\}.*$$

A case of this is easily seen to be

$$V = gr^{-1} + g'r'^{-1} + kR^2,$$

where R is the distance from the centre. (*Vide* Liouville, Vol. 11,

* [Thus $$\int v\, ds = \sqrt{2}\int \sqrt{E - V}\, .\, ds = \sqrt{2}\int \sqrt{(E - V)(\mu^2 - \nu^2)}\, d\sigma$$
$$= \sqrt{2}\int \sqrt{E(\mu^2 - \nu^2) - (f\mu + F\nu)}\, d\sigma,$$
where $d\sigma^2 = d\mu'^2 + d\nu'^2$, and this comes under the first form of § 7.]

1846; or Prof. Cayley's "Report on Special Problems of Dynamics," *Brit. Assoc.*, 1862.)

Further, if $\psi(x, y) = c$ be the equation to the orbit under the law $V = f(x, y)$ when the energy of the motion is E, we see that $\psi(\mu', \nu') = c$ will be the equation to the orbit under the law

$$V = (\mu^2 - \nu^2)^{-1}\{f(\mu', \nu') - E\},$$

where $\mu' = \int (\mu^2 - b^2)^{-\frac{1}{2}} d\mu, \quad \nu' = \int (b^2 - \nu^2)^{-\frac{1}{2}} d\nu,$

the velocity being such that $\frac{1}{2}v^2 + V = 0$.

For instance, in the field,

$$V = A'(\mu^2 - \nu^2)^{-1} = A'(rr')^{-1},$$

particles projected with the velocity from infinity will describe oblique trajectories of the system of confocals.

(2) On taking $\xi = \log r$, $\eta = \theta$, we find $\lambda^2 = r^{-2}$, and we have the solution immediately for the law

$$V = r^{-2}\phi\left(\frac{x}{y}\right) + \psi(r),$$

with *any* velocity of projection, since $\lambda^{-2}\,(= x^2 + y^2)$ is of the proper form.

This law has been discussed by Jacobi and Bertrand from the equations of motion.

(3) Adopting the dipolar system of coordinates

$$\xi = \log r - \log r', \quad \eta = \theta - \theta',$$

we find $\lambda = b(rr')^{-1}$,

and we have at once the equations in quadratures of the orbits in a field of force given by

$$V = \frac{1}{r^2 r'^2}\left\{f\left(\frac{r}{r'}\right) + F\left(\frac{r^2 + r'^2 - b^2}{2rr'}\right)\right\}$$

$$= \frac{1}{r^2 r'^2}\phi\left(\frac{r}{r'}\right) + \psi(rr'), \text{ say,}$$

with the velocity of projection which makes $\frac{1}{2}v^2 + V = 0$.

We may notice that, if $V \propto (rr')^{-2}$, particles projected with velocity from infinity will describe paths cutting all circles through the foci at equal angles.

These results might have been arrived at by the method of inversion.

(4) For motion on an ellipsoid, we have

$$ds^2 = \frac{(\mu^2 - \rho^2)(\mu^2 - \nu^2)}{(\mu^2 - h^2)(\mu^2 - k^2)} d\mu^2 + \frac{(\nu^2 - \mu^2)(\nu^2 - \rho^2)}{(\nu^2 - h^2)(\nu^2 - k^2)} d\nu^2,$$

where ρ, μ, ν are the principal semi-axes of the confocals through the

point, and h^2, k^2 the constant differences of the squares of the semi-axes of each surface. Hence $\int v\,ds$

becomes $$\int v\,(\mu^2-\nu^2)^{\frac{1}{2}}\,(d\theta^2+d\phi^2)^{\frac{1}{2}},$$

where $$\theta = \int \left(\frac{\mu^2-\rho^2}{\mu^2-h^2\,.\,\mu^2-k^2}\right)^{\frac{1}{2}} d\mu,$$

$$\phi = \int \left(\frac{\rho^2-\nu^2}{\nu^2-h^2\,.\,\nu^2-k^2}\right)^{\frac{1}{2}} d\nu.$$

Therefore, if the potential is of the form

$$(\mu^2-\nu^2)^{-1}\{f(\mu)+F(\nu)\},$$

the equation of the trajectory can always be reduced to quadratures, by § 7.* By means of Hamilton's Characteristic equation, Liouville has generalized this result so as to include free motion in space (*Journal*, Vol. 12, 1847).

(5) On a sphere with ordinary polar coordinates,

$$ds^2 = a^2d\theta^2+a^2\sin^2\theta\,.\,d\phi^2.$$

Hence $\int v\,ds$

becomes $$\int a\,v\sin\theta\left\{\left(\frac{d\theta}{\sin\theta}\right)^2+d\phi^2\right\}^{\frac{1}{2}},$$

and similar results may be stated.

This method may clearly be applied very widely; and it possesses the advantage of establishing a complete correspondence between the two systems of orbits, so that, for instance, the stability of either orbit is determined at once when that of the other is known.

8. *A curve of quickest descent* from one point to another in a given field of force, with given velocity of projection from the first point, is such that $$\delta\int\frac{ds}{v}=0,$$

and is therefore the same as a free orbit in a field which would make the velocity equal to v^{-1}; this latter implies a definite law of potential and a definite velocity of projection. The converse proposition is of course equally true. Examples of their application have been given at length by Mr. Townsend in the *Quarterly Journal of Mathematics*, Vol. xv.; this short and direct statement has already been given by Sir W. Thomson (*vide* Tait and Steele's *Dynamics*). We notice that the constrained orbit is not brachistochronous beyond the point at which

* [I find that this method has been applied to the discussion of some cases of motion on an ellipsoid by Mr. W. R. W. Roberts in *Math. Soc. Proc.*, 1883, Vol. xiv., pp. 230—5.]

the free orbit becomes unstable; so that simple observation as to whether a consecutive free orbit does or does not intersect the given one between the two fixed points, in Mr. Townsend's examples, determines at once whether the corresponding brachistochrone is real, or only apparent, as corresponding simply to a maximum-minimum.

9. *For rays of light or sound*, Fermat's Principle of Least Time,

$$\delta \int v^{-1} ds = 0, \quad \text{or} \quad \delta \int \mu \, ds = 0,$$

follows at once from the notion of waves. The analogy of this to the principle of least action for a particle has long been known, and indeed this case of the principle of action appears to have been suggested to Euler by the interpretation of Fermat's principle according to the corpuscular theory of light. (Serret, *Bulletin de Math.*, II., p. 97.) We can therefore apply to the case of rays all the transformations that have been indicated for the motion of a particle, and all the examples already given can be translated into solutions for the paths of rays in heterogeneous media by simply writing μ for v. The restriction as to velocity of projection that often affected the generality of the transformations is here rather an advantage, as it simplifies the law of the index of refraction. The following are simple examples:—

(1) By inverting the straight rays in a uniform medium, we find that, in a medium for which $\mu \propto r^{-2}$, the rays are circles passing through the origin.

(2) If $\mu \propto \sqrt{\frac{2}{r} - \frac{1}{a}}$, the rays are ellipses with the origin as focus, the major axis of each is $2a$, and the rays can never pass outside an exterior caustic which is an ellipse with the radiant point and the origin as foci. (*Vide* Tait and Steele's *Dynamics*, Ch. iv.)

Further, if in these examples r denote the distance from an axis, the particle can have a constant velocity along the axis in addition to the orbital motion round it; and therefore for a medium distributed in cylindric layers round an axis we have for the rays helical curves,

(1) On circular cylinders,

(2) On elliptic cylinders,

whose forms are completely determined. And we see also that the rays in (2) are bounded externally by a caustic elliptic cylinder, and that in (1) they all pass through a line parallel to the axes; and, as the conjugate foci for small pencils lie on the caustics, they are completely determinable, and each pencil has an infinite number of conjugate foci symmetrically placed.

By inverting this class of cases, we arrive at distributions of refractive index which lead to ring-shaped caustics, with corresponding properties.

10. *For a chain of uniform density* hanging in a field of force whose potential is V, the figure of equilibrium is determined by the condition

$$\delta \int (V+\lambda)\, ds = 0,$$

where λ is a constant that has to be adjusted to the given length of the chain, according to the known method of the Calculus of Variations. There is here one condition more than in the case of the motion of a particle. We conclude, however, that the forms assumed by chains of different lengths attached to two fixed points are the trajectories of particles projected from one of the points so as to pass through the other in a field where

$$v = V+\lambda,$$

and the potential is therefore $-\frac{1}{2}(V+\lambda)^2$, λ being variable.

11. When the chain assumes the form of a circle round a centre of force as centre, the determination of its stability is thus at once reduced to Newton's well-known determination of the stability of a circular orbit, by calculation of the apsidal angle. If the apsidal angle be imaginary, the circular form will be stable; if it be real, the circular form will only be stable for arcs shorter than the distance between two consecutive apses. For the condition of stability is that the potential energy be a minimum, and we know, by Jacobi's rule in the Calculus of Variations,* that it can only be a minimum when taken to a limit short of the point in which the curve is met by any consecutive curve of equilibrium of the same length. As was to be expected, the value of the apsidal angle contains λ, and therefore depends on the length of the chain; however, the relation between λ and the radius of the circle is easily determined. In fact, following the analogy with dynamics of a particle, we have, if

$$u = r^{-1},$$

$$\frac{d^2u}{d\theta^2}+u = \frac{P}{h^2u^2} = \frac{V+\lambda}{h^2}\frac{dV}{du},$$

since P, the attractive force in the problem of the particle, is the rate of increase of the potential outwards, and is therefore the rate of decrease of $\frac{1}{2}v^2$ outwards from the origin. When the orbit is nearly circular, we have

$$u = a+x,$$

$$V \equiv \phi u = \phi a + x\phi' a,$$

where x is small; and h is a *constant* for which we may write the

* See Routh's *Rigid Dynamics*, 3rd ed., p. 561.

approximate value $h = vp = (\phi a + \lambda)\, a^{-1}$.

Therefore $$\frac{d^2x}{d\theta^2} + x + a = \frac{ha + x\phi' a}{h^2}(\phi' a + x\phi'' a)$$
$$= \frac{a}{h}\phi' a + x\left\{\left(\frac{\phi' a}{h}\right)^2 + \frac{a\phi'' a}{h}\right\}.$$

Therefore, if we make $\phi' a = h$,

we have for x the oscillatory value

$$x = A \sin(\mu\theta + \alpha),$$

where $$\mu^2 = \frac{a\phi'' a}{\phi' a},$$

and the apsidal angle is π/μ.

For a force varying as the inverse n^{th} power of the distance acting on the string, we have $\phi u = Bu^{n-1}$,

$$\frac{u\phi'' u}{\phi'(u)} = n - 2,$$

so that, if $n < 2$, the apses will be imaginary; and we conclude that the circular form will be unstable for the law of the inverse square, or any higher inverse power.* Under a lower power, a string originally held in a circular or nearly circular form round the centre of force as centre will maintain that form; under any other power, it will depart from that form unless held fast at points whose distance is less than the distance between two consecutive apses. This appears to be at variance with Mr. Townsend's statement (*Quarterly Journal of Math.*, xiii., p. 238).

12. When the force emanates from a fixed axis, the trajectory of a particle will be a curve, on a cylinder whose cross-section is the plane central orbit, such that the velocity parallel to the axis of the cylinder will be constant. When the cross-section is closed, the trajectory will be a helical curve. So also *one* of the catenaries for the corresponding law of force will be this helical curve; but any length more than one turn of it would usually be unstable.

The difficulty in establishing a strict analogy between the orbit of a particle and the form of a catenary arises from the fact that between two points an infinite number of catenaries of different lengths can lie, while only a definite number of orbits with given total energy can pass through them. This explains the appearance of the additional constant λ.

The problem of determining what forces must be applied to a chain,

* [Mr. R. R. Webb has given for the period of that vibration of the string in which it is divided into m complete waves, the expression $\frac{2\pi}{m}\left\{ag\,\frac{m^2+n-2}{m^2+1}\right\}^{-\frac{1}{2}}$, the law of force being $g\,(ar)^{-n}$ (*Cambridge Math. Tripos*, Jan., 1884); which confirms the above statement.]

or how it must be loaded, in order that it may retain a certain given form, is, of course, much less general than the inverse problem of the determination of the system of catenaries that are possible in a given field of force; and accordingly the analogy is easily enough stated for the more special problem, but in that form it applies only to the special curve considered.

The following, however, is a specimen of the kind of results that flow from the more general form of the analogy :—

Suppose rays diverging from a centre in a medium of varying index μ to touch a caustic; the paths of the rays will be the curves assumed by uniform strings, fastened to the fixed centre and wrapped round the caustic, in a field of force whose potential is equal to μ.

The provision that the string has to touch the caustic, here disposes of the extra degree of generality in the catenaries, and limits us to that *one* which passes through the two terminals, and has the same tangent as the ray at one of them.

13. A very interesting *series of natural catenaries* is included in the problem of finding the form assumed by a flexible string carrying an electric current in a given magnetic field. Here the energy condition leads to the solution that for equilibrium the variation of the number of tubes of magnetic force enclosed by the string should be zero, while for stability the number of tubes enclosed must be a maximum.

It follows that all such catenaries are geodesic curves on the surfaces formed by the lines of force which intersect them; for, if not, then, by making them geodesics on wider surfaces, they can be made to include more lines of force.

Let us take the case in which the force is everywhere in a fixed direction. We know that it must then also be uniform. The catenary will be a geodesic on a cylindrical surface, and would therefore become straight when the surface is unrolled. Hence its projection on a cross-section of the surface will also be of constant length, and the cylinder will therefore include the greatest number of tubes of force (*i.e.*, the greatest area) when it is circular. Thus the catenaries will all be circular helices; but it is clear that more than a single turn of the helix will be unstable, unless the string is confined by being enclosed in a non-conducting cylinder, or wrapped round one. Even if this cylinder do not lie along the lines of force, or if it is not circular, the portion of string in contact with it will still lie along a geodesic.

Le Roux has observed the helical form in a platinum wire, rendered flexible by incandescence, and placed in the sensibly uniform magnetic field between two flat poles of an electro-magnet. Professor Stokes has deduced the helical form analytically in connection with the

explanation of the spiral discharges in a rarefied gas, along a magnetic field, observed by Spottiswoode. (*Vide* Chrystal, *Encyc. Brit.*, *Art.* "*Electricity*"; Spottiswoode, *Proc. Roy. Soc.*, 1875.) We notice that, to explain the number of convolutions observed in the spirals, we must also take cognizance of the outer and less heated portions of the gas as performing the part of the constraining cylinder mentioned above.

The method which we have just employed shows also that, in a field of force radiating from a centre, the catenary will be a geodesic on a right circular cone; while, to obtain more than a single turn of the spiral, the string must be enclosed in or wrapped round a circular cone. This form of the catenary has been already deduced analytically by Darboux.

A flexible electric current attracted by a very powerful straight current will, if lying in one plane, assume a form such that

$$\cos\psi = 1 - A\log\frac{y}{a},$$

where ψ is the angle its direction makes with that of the straight current, at a point whose distance from it is y.

On the Direct Application of the Principle of Least Action to the Dynamics of Solid and Fluid Systems, and Analogous Elastic Problems. *By* Professor J. Larmor, Queen's College, Galway.

[*Read March 13th*, 1884.]

1. In the preceding paper it has been observed that the most direct and compendious method of stating the mathematical conditions of a physical problem is to express it as a maximum or minimum relation, and that in dynamics this can always be accomplished by means of the Principle of Least Action. It is natural, then, to employ this principle in the transformation of dynamical problems, and in the investigation of analogies between different departments of the science; and it was so employed in connection with Dynamics of a Particle and related subjects.

We proceed to apply the same method to dynamics of general systems. There are two well-known forms of the minimum principle which we shall use, viz., that if T be the kinetic energy, and V the potential energy of the system, then, between any two fixed configurations,

(1) $\delta\int(T-V)\,dt = 0$, when the time of motion is constant,

(2) $\delta\int T dt = 0$, when the total energy of the motion $E\,(\equiv T+V)$ is constant.

The second is the more usual form of the principle, and admits of direct elementary proof (*cf.* Thomson and Tait's *Nat. Phil.*, § 327). The first, which is more immediately connected with the Lagrangian equations of motion, can be even more easily established. For, using Cartesian coordinates, we are to prove that

$$\delta \int \{\tfrac{1}{2}\Sigma m\,(\dot{x}^2+\dot{y}^2+\dot{z}^2)-V\}\,dt = 0.$$

But with fixed initial and final configurations, and with the time limits of the integral fixed, this variation is, by the ordinary rule, when we vary the coordinates and not the time, equal to

$$\int \{-\Sigma m\,(\ddot{x}\,\delta x+\ddot{y}\,\delta y+\ddot{z}\,\delta z)-\delta V\}\,dt;$$

and the quantity inside the integral, being the virtual work of the applied forces opposed by the effective forces, is equal to zero, by D'Alembert's principle and the Principle of Virtual Velocities.

When the time of the motion is not supposed constant, we must vary both the coordinates and t: the process is not much more difficult (*vide* Routh's *Rigid Dynamics*, 3rd ed., foot-note, p. 307), and shows that the variation of the time of passage between the fixed configurations introduces simply a term $-E\delta t$. In fact, the complete variation is

$$[(T-V)\,\delta t+\Sigma m\,\{\dot{x}\,(\delta x-\dot{x}\,\delta t)+\ldots+\ldots\}]$$
$$+\int\Big[-\Sigma m\,\{\ddot{x}\,(\delta x-\dot{x}\,\delta t)+\ldots+\ldots\}-\Big\{\frac{dV}{dx}\,(\delta x-\dot{x}\,\delta t)+\ldots+\ldots\Big\}\Big]\,dt,$$

since $\delta x-\dot{x}\,\delta t$, and not δx, is now the virtual displacement of the particle. In this expression the first term refers to the limits, at which δx, δy, δz are zero. The integral term is zero as before. Hence, if δt represent the variation of the time of passage, we have

$$\delta \int (T-V)\,dt = -E\,\delta t.$$

Again, if we now suppose that E is constant during the motion, we have

$$\delta \int (T+V)\,dt = \delta\,(Et) = E\,\delta t+t\delta E;$$

therefore, adding,

$$2\delta \int T\,dt = t\delta E,$$

so that, if the energy of the motion is not varied,

$$\delta \int T\,dt = 0,$$

which is the second form of the principle.

Having once shown the truth of the first form, of course it only remains to transform T and V into any generalized coordinates, and the method of variation will then afford a proof of Lagrange's equations of motion.

2. In treating of the motion of a general system, it is important to find the forms assumed by these principles in the cases in which some of the coordinates do not explicitly appear in the expressions for the

kinetic and potential energies, and in which the corresponding momenta are therefore constant. From this modified form the equations of motion with ignored coordinates, as developed independently by Routh and Sir W. Thomson, are immediate deductions.

Kirchhoff's analogy between the form assumed by an elastic wire originally straight, when subjected to stresses at its ends, and the motion of a solid about a fixed point under gravity, is explained from this fundamental point of view; and an extension to the case in which the wire was originally of circular or spiral form immediately suggests itself. The brachistochronous character of the motion of a sphere in a fluid of the same density with any fixed boundaries or obstacles whatever, is pointed out.

The principle is then applied to the motion of solids in a fluid with cyclosis through apertures either in some of the solids or in fixed boundaries. A theorem of Kirchhoff's relating to circulation of fluid round thin rigid cores is proved immediately, and equations are deduced for the motion of the solids which, I have since found, agree with those given as the result of a much longer analytical process by Sir W. Thomson in 1872. They are also consistent with the result of a long direct hydrodynamical investigation by Dr. C. Neumann, published in 1883.

3. In the very important case in which the expressions for the potential and kinetic energies do not contain some of the coordinates the corresponding momenta are constant, and therefore, by elimination, the velocities can also be made to disappear from the expressions, being replaced by these momenta. Lagrange's and Hamilton's equations of motion have been independently adapted to this case by Routh and Sir W. Thomson, and applied to very important problems. When the variables are reduced in number by means of these equations of condition, the original functions of Hamilton no longer retain their minimum properties, and we shall investigate the functions to which they now belong. Following Routh's example, we shall at first proceed as if the momenta introduced were not necessarily constant.

The coordinates are thus divided into two groups; one group of velocities is retained, while the other is eliminated, being replaced by the corresponding group of momenta.

Let T, V (the kinetic and potential energies of the motion) be expressed in terms of a set of coordinates θ, ... and another set ψ, ..., and suppose we wish to obtain a form of the principle that shall involve the corresponding momenta Ψ, ... instead of the velocities $\dot{\psi}$,... If $L = T - V$, the first form of the principle is that

$$\delta S \equiv \delta \int L \, dt = 0,$$

with time of motion given.

Now $$\delta \int L\,dt = \int \left(\frac{dL}{d\theta}\delta\theta + \frac{dL}{d\dot\theta}\delta\dot\theta + \ldots + \frac{dL}{d\psi}\delta\psi + \frac{dL}{d\dot\psi}\delta\dot\psi + \ldots\right) dt,$$

a variation of the time being unnecessary. In this we are to write Ψ for $\frac{dL}{d\dot\psi}$, ..., and to substitute for $\dot\psi$, $\delta\dot\psi$, ... from these relations. We find

$$\int \left(\frac{dL}{d\theta}\delta\theta + \frac{dL}{d\dot\theta}\delta\dot\theta + \ldots\right) dt$$

$$= \delta \int (L - \Psi\dot\psi - \ldots)\,dt + \int \left(\dot\psi\delta\Psi - \frac{dL}{d\psi}\,d\psi + \ldots\right) dt.$$

But the first side of this identity is the variation of S produced by varying θ, $\dot\theta$, ... without causing any variation in ψ, $\dot\psi$, ..., and is therefore zero. Hence, with our new coordinates θ, $\dot\theta$, ..., ψ, Ψ, ... the second side is zero.

We shall write $$L' = L - \Psi\dot\psi - \ldots .$$

Since $$L = T - V = \tfrac{1}{2}(\Theta\dot\theta + \ldots + \Psi\dot\psi + \ldots) - V,$$

we see that L' differs from L by containing the difference between the part of the kinetic energy due to the momenta Θ, ... and the part due to the momenta Ψ, ... instead of the whole kinetic energy, and by being expressed in terms of different variables. We shall find that this observation will often very much simplify the determination of L', especially in cases of cyclical fluid motion.*

In case the ψ coordinates do not enter into T or V, we have Ψ, ... constants; and the variational equation of motion becomes simply

$$\delta S' \equiv \delta \int L'dt = 0,$$

with time of motion given. Therefore, in such steady motion, $\int L'\,dt$ is a minimum, for the given values of the steady momenta, and the given value of the time of motion, provided, as usual, that the interval be not so long as to make it only stationary.

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* [The total quantity of work required to impulsively generate the motion is independent of the order of operations; but the quantity consumed by each impulse depends on the order of succession in which the impulses are applied. The expressions in the text refer to the case in which all the impulses are simultaneously and equably applied to the system, so that the state of motion rises from zero to its final value, remaining always similar to itself. In the example of § 10, the motion is therefore to be supposed generated by the simultaneous and equable application of Θ, ... to the boundaries of the fluid, and Ψ, ... to the barriers.

If, however, the gyrostats possess initially their steady constant momenta Ψ, ..., the impulses that must be applied to the solids to which they are attached in order to start their motion are equal to $\frac{dL'}{d\dot\theta}$, ... respectively; as is found by taking the time integral of the modified equations of motion of this section, extended over the duration of the impulse. Thus we see that, in the case of § 10, the same impulses are required to start the solids, whether the fluid be circulating or not. But the amount of work required is different, being $\frac{1}{2}\rho\int(\phi_0+\phi_1)\frac{d\phi_0}{dn}\,d\mathrm{S}$ in the former, and $\frac{1}{2}\rho\int\phi_0\frac{d\phi_0}{dn}\,d\mathrm{S}$ in the latter case.—*Sept.* 4.]

The equations of motion are therefore, by the Calculus of Variations,

$$\frac{d}{dt}\frac{dL'}{d\dot\theta} - \frac{dL'}{d\theta} = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

and are exactly analogous to Lagrange's equations, except that L' now contains terms with $\dot\theta, \ldots$ in the first degree.

If the motion is not steady, so that $\Psi, \ldots$ are not constants, we have the additional equations

$$\frac{dL'}{d\Psi} = -\dot\psi, \ldots,$$

$$\frac{dL'}{d\psi} = \frac{dL}{d\psi}, \ldots .$$

These two sets of equations, the former of the Lagrangian, and the latter of the Hamiltonian type, were given by Routh under the name of modified Lagrangian equations.

4. The other form of the variational equation, $\delta A = 0$, with the total energy E of the motion given, where $A = \int T\,dt$, may be reduced to the new variables as follows. The time is now variable, and therefore, as usual, $\delta \int L\,dt = [L\delta t]$

$$+\int\left(\frac{dL}{d\theta}(\delta\theta - \theta\,\delta t) + \frac{dL}{d\dot\theta}(\delta\dot\theta - \ddot\theta\,\delta t) + \ldots + \frac{dL}{d\dot\psi}(\delta\dot\psi - \ddot\psi\,\delta t) + \ldots\right) dt,$$

as we are now, for simplicity, taking $\Psi, \ldots$ constant. Therefore

$$\delta \int L'dt = [L'\delta t] + \left[\Sigma\,\frac{dT}{d\dot\theta}(\delta\theta - \dot\theta\,\delta t)\right],$$

by the usual method of variation, as the terms under the integral vanish;

$$= [L'\delta t] + \left[\Sigma\,\frac{dT}{d\dot\theta}\,\delta\theta\right] - [(2T - \Psi\dot\psi - \ldots)\,\delta t]$$

$$= -[E\delta t] + \left[\Sigma\,\frac{dT}{d\dot\theta}\,\delta\theta\right].$$

But $E = T + V,$

therefore $\delta \int (T+V)\,dt = E\,\delta t + t\delta E,$

and, adding this to the above result, we find

$$\delta \int (2T - \Psi\dot\psi - \ldots)\,dt = t\delta E + \left[\Sigma\,\frac{dT}{d\dot\theta}\,\delta\theta\right].$$

We conclude that $\delta A' \equiv \delta \int T'dt = 0,$

with given energy, where,

$$T' = T - \tfrac{1}{2}(\Psi\dot\psi + \ldots),$$

i.e., where T' is equal to the part of the kinetic energy which is due to the momenta $\Theta, \ldots$, but not to $\Psi, \ldots$.

5. The values of the quantities under the integral signs in these modified formulæ can be easily calculated. Thus, following Routh (*Essay on Stability*, Ch. iv., Art. 21), let us write

$$T \equiv \tfrac{1}{2}T_{\theta\theta}\dot{\theta}^2 + T_{\theta\phi}\dot{\theta}\dot{\phi} + \ldots + \tfrac{1}{2}T_{\psi\psi}\dot{\psi}^2 + T_{\psi\chi}\dot{\psi}\dot{\chi} + \ldots,$$

and we have

$$T_{\psi\psi}\dot{\psi} + T_{\psi\chi}\dot{\chi} + \ldots = \Psi - T_{\psi\theta}\dot{\theta} - T_{\psi\phi}\dot{\phi} - \ldots,$$

$$T_{\psi\chi}\dot{\psi} + T_{\chi\chi}\dot{\chi} + \ldots = \mathrm{X} - T_{\chi\theta}\dot{\theta} - T_{\chi\phi}\dot{\phi} - \ldots,$$

$$\ldots = \ldots$$

where we shall, for brevity, write the second sides

$$\Psi - X,\ \mathrm{X} - Y,\ \ldots.$$

Now

$$T = \tfrac{1}{2}T_{\theta\theta}\dot{\theta}^2 + T_{\theta\phi}\dot{\theta}\dot{\phi} + \ldots + \tfrac{1}{2}\dot{\psi}(\Psi + X) + \tfrac{1}{2}\dot{\chi}(\mathrm{X} + Y) + \ldots$$

$$L' = \tfrac{1}{2}T_{\theta\theta}\dot{\theta}^2 + T_{\theta\phi}\dot{\theta}\dot{\phi} + \ldots - V - \tfrac{1}{2}\dot{\psi}(\Psi - X) - \tfrac{1}{2}\dot{\chi}(\mathrm{X} - Y) - \ldots$$

$$T' = \tfrac{1}{2}T_{\theta\theta}\dot{\theta}^2 + T_{\theta\phi}\dot{\theta}\dot{\phi} + \ldots + \tfrac{1}{2}\dot{\psi}X + \tfrac{1}{2}\dot{\chi}Y + \ldots;$$

and, eliminating $\dot{\psi}$, $\dot{\chi}$, ... from each of these and the above set by a determinant, we find

$$T = \tfrac{1}{2}T_{\theta\theta}\theta^2 + \ldots - \frac{1}{2\Delta}\begin{vmatrix} 0, & \Psi + X, & \mathrm{X} + Y & \ldots \\ \Psi - X, & T_{\psi\psi}, & T_{\psi\chi} & \ldots \\ \mathrm{X} - Y, & T_{\psi\chi}, & T_{\chi\chi} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

in which the determinant does not contain any terms in Ψ, X, ... of the first degree, as it is not altered by changing the signs of these quantities:

$L' = \tfrac{1}{2}T_{\theta\theta}\dot{\theta}^2 + \ldots - V +$ a symmetric determinant of the same kind, which does contain terms of the form $\dot{\theta}\Psi$, $\dot{\theta}\mathrm{X}$, ...:

$$T' = \tfrac{1}{2}T_{\theta\theta}\dot{\theta}^2 + \ldots - \frac{1}{2\Delta}\begin{vmatrix} 0, & X, & Y, & \ldots \\ \Psi - X, & T_{\psi\psi}, & T_{\psi\chi}, & \ldots \\ \mathrm{X} - Y, & T_{\psi\chi}, & T_{\chi\chi}, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

which is the same as L' with the omission of V, of the quadratic terms in Ψ, X ..., and of half of each term of the first degree in $\dot{\theta}$, $\dot{\phi}$ ....

An illustration of these is Sir W. Thomson's expression that will be proved below independently for the kinetic energy in the case of solids in motion in a cyclically moving fluid, and the associated functions L' and T' for that problem.

6. As a first example of the application of the method, we will take Kirchhoff's analogy between the form of a bent elastic wire and the motion of a solid body round a fixed point.

If we consider a naturally straight elastic wire of any form of section, provided it be small and uniform, we can express its potential

energy of deformation by the following known plan (Thomson and Tait's *Nat. Phil.*, Arts. 593—97) :—Draw through each point of the central axis of the wire three rectangular axes in fixed directions in space, one of them in the direction of its length, and suppose them to be rigidly connected with the substance of the wire ; then, when it is deformed, the directions of these axes will be altered, and the deformation at any point will be completely specified by the displacements of their directions per unit length measured along the wire ; and, if the curvature be not so great as to prevent our assuming the stresses to be proportional to the strains produced by them, the potential energy of deformation per unit length will, by Green's theory of elastic solids, be a homogeneous quadratic function of these rates of variation of direction of the axes. That is, if we suppose a point to travel along the wire with unit velocity, and suppose a certain solid to rotate round a fixed point, so that three axes fixed in the solid shall be always parallel to the axes through the point moving along the wire, the potential energy of deformation per unit length will be equal to the kinetic energy of rotation of this solid.

Let one end of the wire be clamped, and the other end be acted on by a given system of forces, reducible to a force and a couple in the ordinary way. The total potential energy of the wire will assume the form of a line integral along the central section, if we replace the balancing stresses at its ends by equal balancing stresses at the ends of each element of it ; for the virtual moment of the balancing couples due to a rigid displacement of the element will be zero, while that of the forces will be $P\,ds \sin\theta\,d\theta$, wherein P is the intensity of the force, and θ is the angle which ds makes with the constant direction of P. Hence the energy condition of equilibrium will be

$$\delta V \equiv \delta \int (T + P\cos\theta)\,ds = 0,$$

between the given terminal points, and with the *given constant length of the wire*, where T is the quadratic function before referred to.

But, for the rotating body of the analogy, the equations of motion may be expressed as the conditions that

$$\delta L \equiv \delta \int (T - V')\,dt = 0,$$

between the given initial and final configurations, with *constant time of motion*.

The two conditions will agree exactly, if

$$V' = -P\cos\theta,$$

and this is clearly secured by causing the constant force P to act on the solid in its constant direction at a point unit distance along that axis which is parallel to the direction of the wire. Kirchhoff's pro-

position follows, then, as an immediate consequence, that the motion of the solid round a fixed point under gravity bears to the deformation of the wire the relation already explained.

7. The analogy does not in this case throw much light on the question of stability, but we may discuss it by considerations similar to those employed in the case of a chain. For the stability of the wire it is necessary and sufficient that V should be an absolute minimum with the given conditions at the extremities, and the given length. But Jacobi's theorem before quoted shows that V is a minimum only up to the first point at which the curve is cut by a neighbouring curve satisfying the conditions. If the curve be of sinuous form and the wire only tied at the ends, it is therefore clear that it will not be stable if longer than half a wave-length, unless it be tied down at points within that distance; for the curve is intersected by an equal curve at points whose distance apart is half a wave-length. For a steel band, clamped at the ends, the limits are wider, but do not apparently amount to three-quarters of a wave, even in the comparatively favourable case in which it is confined to one plane, and therefore assumes one of the forms corresponding to the simple pendulum (figured in Thomson and Tait, and in Rankine's *Applied Mechanics*); for that case, however, the instability is practically removed by the passive friction of the plane surface on which the curve is formed, unless the displacement be so great that the disturbing force exceeds the frictional limits.

8. If the elastic wire, instead of being straight, had been originally a helix, the three components of whose curvature are each constant, we should have had for the potential energy of bending, per unit length, a quadratic function of the component curvatures, each diminished by the corresponding constant, and the energy condition of equilibrium might be written down as before. We notice that it now contains terms of the first degree with constant coefficients in the expression for T; but these are exactly such terms as would be introduced into the modified T of the dynamical problem by fly-wheels mounted on the rotating body. So that we are prepared for this extension of Kirchhoff's proposition, viz. — That the motion of a gyroscopic pendulum (*i.e.*, a solid rotating about a fixed point with a revolving fly-wheel mounted on an axis fixed in it) is exactly analogous to the form assumed by a wire originally helical, when deformed by any distribution of forces applied to its extremities. Indeed, an independent proof is now obvious; for the constant angular momentum of the fly-wheel is equivalent to three constant component angular momenta round the axes through the point of suspension fixed in the solid, in addition to the angular momenta that

are due to the action of the forces, while in the wire there are the three constant component curvatures round the three axes, in addition to those caused by the action of the forces; and, as for the principal axes of the solid angular momenta round them are proportional to the angular velocities round them, and these latter are equal to the curvatures in the wire, we see that the analogy still holds.

The solid of the analogy may also be replaced by a pendulum having annular cavities filled with fluid which is circulating round them, or by a pendulum in a fluid with apertures in it, through which the fluid is circulating; as the motion of each of these follows the same laws as that of the gyroscopic pendulum.

A wire of uniform circular section corresponds to a solid symmetrical about an axis; and the fly-wheel is to be attached with its centre of gravity in the axis, and with its axis in such a direction that its component angular momenta round the axis and in a perpendicular direction are equal to the couples corresponding to the tortuosity and curvature of the helix, respectively. A helix with a few convolutions far apart corresponds to a slowly rotating fly-wheel, while a helix with many close convolutions corresponds to a rapidly rotating wheel, mounted nearly parallel to the axis, and the forms which it assumes represent the motion of the solid with this "gyrostatic domination." Under these latter circumstances, the helix bends and twists very much as if it were a straight wire; so that the solid moves with its axis in rapid gyration nearly at right angles to a mean position whose own motion does not very greatly differ in character from that of a solid moving freely round the fixed point under the same forces.

As a still simpler illustration of this extension, we notice that a spring of the form of a circular arc will, when deformed by end forces, in its own plane, assume one of the known forms of the elastic curve for a straight spring; and from such a wire we might obtain the curves due to a distribution of force *and couple* on each extremity of a straight spring, without having to apply the bending force at the end of a rigid arm attached to the spring. (*cf.* Thomson and Tait.)

Again, such a spring when bent out of one plane will correspond to the motion of a rotating body with a gyrostat attached to it, having its axis parallel to one of the principal axes through the point of suspension.

9. The solution of questions relating to the motion of solid bodies in fluids without cyclosis has been reduced to Lagrange's equations in Thomson and Tait (2nd ed., Art. 320). In this case the ignored coordinates are those corresponding to the individual particles of fluid, and the momenta that correspond to them are the momenta remaining in the fluid when the solids have been all reduced to rest. In the

problem discussed by Thomson and Tait, these momenta are all zero; but, when the fluid circulates through apertures in the solids, the momenta are definite, and are linear functions of the cyclic constants of all the independent circuits.

Recalling the argument already employed, which is simplified now by the constancy of these momenta, we have

$$\delta S \equiv \delta \int L\,dt = \int \left(\frac{dL}{d\theta}\,d\theta + \frac{dL}{d\dot{\theta}}\,\delta\dot{\theta} + \ldots + \Psi\,\delta\dot{\psi} + \ldots \right) dt,$$

with given time of motion, and therefore

$$\delta S' \equiv \delta \int (L - \Psi\dot{\psi} - \ldots)\,dt = \int \left(\frac{dL}{d\theta}\,\delta\theta + \frac{dL}{d\dot{\theta}}\,\delta\dot{\theta} + \ldots \right) dt.$$

That is, the total variation of S' when expressed in terms of θ, ..., $\dot{\theta}$, ..., and the constants Ψ, ..., is equal to that part of the variation of S which is due to variations of θ, ..., $\dot{\theta}$, ... alone, while $\dot{\psi}$, ... remain constant; it is therefore zero. And, also as before, we see that

$$S' \equiv \int L'\,dt,$$

where L' is obtained by subtracting from the total energy of the system twice that part which is due to the constant momenta.

Let us consider, however, in the first place, the case in which there is no cyclosis, and therefore Ψ, ... are each zero. We have then for δS the same value as if there were no ignored coordinates; and the ordinary proof for rigid systems (§ 1) now suffices to deduce the other minimum theorem, that for given total energy of motion

$$A \equiv \int T\,dt,$$

satisfies the minimum condition, between any two configurations of the motion.

For example, suppose we have a solid sphere moving in fluid enclosed in a boundary of any shape, and with any fixed obstacles immersed, but without circulation. The only coordinates that we are concerned with are those of the centre of the sphere; and T and E can be expressed in terms of them. But, if further, the sphere be of the same density as the fluid, or if no external forces act on the sphere, we have T constant, and the variational equation becomes $\delta \int dt = 0$ with given energy. We conclude that the sphere will move from one point to another in a shorter time than if it were guided by frictionless constraint so as to proceed by any other path, it being understood, as usual, that for paths beyond a certain length the minimum relation may cease to hold, and the time may be only stationary. The same proposition applies to cylinders moving in two

dimensions. It follows, just as in the case of motion of a particle, that if equal spheres are projected from a point with equal energies in variable directions, their paths are everywhere intersected at right angles by a system of surfaces of equal action, which are here also surfaces of equal time.

10. Now let us discuss the case of perforated bodies with circulation through their apertures. The kinetic energy T of the motion has been shown by Sir W. Thomson to consist of a quadratic function $\mathfrak{T}$ of the component velocities of the solids, the method of calculating which is not affected by the existence of cyclosis, together with a quadratic function Ω of the cyclic constants $\kappa_1, \kappa_2 \ldots$, which latter clearly represents the kinetic energy of the motion that would remain after the solids are brought to rest. (*Vide* Lamb's *Fluid Motion*, Art. 120.) The momenta due to the ignored coordinates are those which contain $\kappa_1, \kappa_2 \ldots$; and the part of the kinetic energy due to these momenta (*i.e.*, the work that would be done by the corresponding impulses in the instantaneous production from rest of the actual fluid motion, the amount of which is obtained at once by integrating over each barrier closing an aperture half the product of impulsive pressure $\kappa\rho$ and velocity $d\phi/dn$) is

$$\tfrac{1}{2}\rho\Sigma\int \kappa \frac{d\phi}{dn}\, d\sigma,$$

taken over the surfaces of all the barriers. (We draw dn outwards from a boundary, and in the direction of the circulation from a barrier.) But this expression is equal to the above-mentioned quadratic function Ω of $\kappa_1, \kappa_2 \ldots$, together with terms involving products of κ's, and velocity components of the solids; and these latter are exactly equal to

$$-\tfrac{1}{2}\rho\Sigma\int \phi_1 \frac{d\phi}{dn}\, dS,$$

taken over the surfaces of the immersed solids, where ϕ_1 is the part of the velocity potential due to cyclosis, and therefore $d\phi_1/dn$ is zero. These statements indeed all follow at once from Green's theorem, which can be applied to each of the acyclic compartments into which the whole space is divided by the barriers; thus, if ϕ_0 is the part due to the motion of the solids, so that $\phi = \phi_0 + \phi_1$,

$$\text{kinetic energy of fluid} = \tfrac{1}{2}\rho\int\left\{\left(\frac{d\varphi}{dx}\right)^2+\left(\frac{d\phi}{dy}\right)^2+\left(\frac{d\phi}{dz}\right)^2\right\}dv$$

$$= \tfrac{1}{2}\rho\left\{\Sigma\int\varphi\frac{d\phi}{dn}\,dS+\Sigma\int\kappa\frac{d\phi}{dn}\,d\sigma\right\};$$

but, for the whole surface of each of the compartments, we have

$$\int \left(\phi_1 \frac{d\phi_0}{dn} - \phi_0 \frac{d\phi_1}{dn}\right) dS = \int (\phi_1 \nabla^2 \phi_0 - \phi_0 \nabla^2 \phi_1)\, dv = 0,$$

and therefore, adding for all the compartments,

$$\Sigma \int \phi_1 \frac{d\phi_0}{dn} dS = 0;$$

or, in our previous notation,

$$\Sigma \int \kappa \frac{d\phi_0}{dn} d\sigma + \Sigma \int \phi_1 \frac{d\phi}{dn} dS = 0.$$

We have, therefore,

kinetic energy of fluid

$$= \tfrac{1}{2}\rho\Sigma \int \phi_0 \frac{d\phi_0}{dn} dS + \text{the quadratic function } \Omega \text{ of } \kappa_1 \kappa_2 \ldots,$$

in which the coefficients of the quadratics are all functions of the coordinates.

The modified principal function is therefore

$$S' = \int L'\, dt,$$

where
$$L' = T_0 + \tfrac{1}{2}\rho \left\{ \Sigma \int \phi_0 \frac{d\phi_0}{dn} dS + 2\Sigma \int \phi_1 \frac{d\phi_0}{dn} dS \right\} - \Omega - V$$

$$= T_0 + \tfrac{1}{2}\rho\Sigma \int (\phi_0 + 2\phi_1) \frac{d\phi}{dn} dS - \Omega - V,$$

and T_0 is the kinetic energy of the solids by themselves. So that L' is now a non-homogeneous quadratic function in $\dot{\theta}, \ldots, \kappa_1, \ldots$ Its differential coefficient with respect to κ_r is the flux through the corresponding aperture with sign changed; its differential coefficients with respect to the velocities $\dot{\theta}, \ldots$ are the impulses required to start the motion of the solids in the circulating fluid.

It is obvious that these expressions are cases of the general expressions for L and L' which have been calculated above (§ 5) for dynamical systems.

The equations of motion are of the form

$$\frac{d}{dt}\frac{dL'}{d\dot{\theta}} - \frac{dL'}{d\theta} = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

and the determination of the motion is thus reduced to analysis.

11. If we suppose the immersed solids to be mere rings or cores of extreme thinness, round which the fluid circulates, while they offer no obstacles to its other motion, the terms of the first degree in L'

will disappear, and we have

$$L' = T_0 - \Omega - V.$$

In other words, L' is the same as if there were no circulation, but the potential of the field in which the cores are placed were increased by Ω. Now Helmholtz has shown in his Memoir on "Vortex Motion" that Ω is equal to the potential energy of electric currents, whose strengths are equal to the respective cyclic constants multiplied by ρ, and which circulate round the cores. We arrive, therefore, at the result that such cores in a fluid appear to exert forces on one another which are the same as would be exerted if they were carrying these electric currents, *i.e.*, that forces equal and opposite to these would have to be applied in order to prevent their motion. The proof of this result for two cores is the subject of a paper by Kirchhoff (*Borchardt*, Bd. 71, 1869; *Gesamm. Abhandlungen*, pp. 404—16), who reduces the potential of the forces to Neumann's well-known form of the potential energy of two electric currents.

12. The general equations of motion that we have obtained are in accordance with equations deduced from the fundamental equations of hydrodynamics by Dr. Carl Neumann (*Hydrodynamische Untersuchungen*, 1883, pp. 1—90). He gives them as a correction to those in Thomson and Tait; but he appears to have misunderstood the reasoning and the limitations there given, his account of it being merely taken from a short notice in Hicks' Report on Hydrodynamics (*Brit. Assoc. Report*, 1882).

Since developing the form of solution just given, I have found that the same problem has been discussed by Sir W. Thomson himself, whose final equations agree with those here given (*Proc. Roy. Soc. Edin.*, 1871—72; *Phil. Mag.*, May, 1873). His method there consists in forming the generalized Hamiltonian equations of motion in terms of the momenta and coordinates of the system (which are clearly exactly suited to the problem), and transforming them analytically into the Lagrangian form; though at bottom the same, the argument is perhaps hardly so compact as that which we have chosen.

Sir W. Thomson proceeds to consider the case in which there are moving solids as well as cores in the fluid. When there is a moving sphere, he points out that the additional terms in T are $\frac{1}{2}\mu q^2 + w$, where q is the velocity that the fluid would have if the sphere were absent, μ is its mass increased by half that of the fluid displaced, and w is the kinetic energy that would be destroyed if the fluid which would then occupy its position were solidified by the action of internal forces. When the sphere is very small, the latter part may be neglected. He applies this result to determine the conditions on which it depends

whether a small spherule of the same density as the fluid will be sucked into the axis of a straight vortex, or will escape from it, and shows that its path is a Cotes' Spiral.

Still neglecting w, it is clear that the forces which the cyclically moving fluid appears to exert on the sphere are the same as the corresponding electric currents would exert on a magnetic molecule with its axis pointing in the direction of the velocity q, and whose moment is $3q\rho/8\pi$ multiplied by the volume of the sphere; for, as regards the velocity produced by it at a point outside it, the sphere may be replaced by the small core which is the analogue in Kirchhoff's theorem of this magnetic molecule.

13. This method may also be applied to prove Kirchhoff's theorem itself. For each core may be replaced by a system of indefinitely small cores, forming a network, just as Ampère similarly subdivided a finite electric circuit. The mutual force exerted by two of these elementary cores is the same as if each was replaced by the corresponding sphere, as above; for the force which one exerts on the other is the effect of the motion which it propagates through the fluid, and at any distance great compared with the radius this motion is the same for the sphere as for the core. The resultant force exerted on either sphere can easily be found by integrating the pressures on different parts of it; and it follows that a force equal and opposite to this must be applied to keep the sphere from moving. This will be found to agree with Kirchhoff's rule: which can now be extended to the general case by summing up for all the elementary cores.

14. The employment of the modified form of the Principle of Least Action, instead of the Principal Function, leads us to a minimum theorem, which is rather simpler than that given above, but is not so easily transformed into equations of motion. It is as follows:—

Of all the motions between fixed initial and final configurations that have the same total energy and the same cyclic constants, the actual free motions are those that possess the property that $\delta A' \equiv \delta \int T' dt$ is zero, where T′ is equal to the kinetic energy of the immersed solids together with

$$\tfrac{1}{2}\rho \int \phi \frac{d\phi}{dn}\, dS,$$

taken over their surfaces; or, in other words, T' is equal to the amount of work that must be properly applied at any instant to the solids [in the manner specified above, § 3, note] in order to start the actual motion. This integral $\int T' dt$ is therefore smaller for the actual motion than for any other neighbouring motion that might be produced in the same cyclically moving fluid by the introduction of frictionless constraints, which do not alter the energy.

[15. The application of Lagrange's equations to the motion of solids in a fluid without circulation, by Thomson and Tait in 1867, was justified first by Kirchhoff (*Borchardt*, 1870); his method was extended somewhat by Boltzmann (*Borchardt*, 1871), and appears from him in the German edition of Thomson and Tait. Another method was used independently by J. Purser (*Phil. Mag.*, 1876).

The method of Kirchhoff admits of easy extension to the general theory of § 10. Neumann considered the question in this way. Thus, by § 1, we have, if we vary the limits, but not the time,

$$\delta \int_{t_0}^{t_1} L\,dt = \left[\Sigma m \left(\frac{dx}{dt}\,\delta x + \frac{dy}{dt}\,\delta y + \frac{dz}{dt}\,\delta z\right)\right]$$
$$= \left[\rho \int \left(\frac{d\phi}{dx}\,\delta x + \frac{d\phi}{dy}\,\delta y + \frac{d\phi}{dz}\,\delta z\right)\right]$$

taken at the limits t_0 and t_1.

Here δx, δy, δz are any displacements; let us now take them to be consistent with a velocity potential, and therefore equal to $\frac{d\,\delta\phi}{dx}$, $\frac{d\,\delta\phi}{dy}$, $\frac{d\,\delta\phi}{dz}$, and we have, by Green's theorem,

$$\delta \int L\,dt = \left[\rho \int \phi\,\frac{d\,\delta\phi}{dn}\,dS\right],$$

the integral being extended over all the boundaries and barriers. If now we suppose the solids not to be displaced, $d\,\delta\phi/dn$ is zero at each boundary. Hence

(1) if there is no circulation in the fluid, $\delta \int L\,dt = 0$ for given initial and terminal position of the solids, and given time of motion; and Lagrange's equations of motion therefore apply:

(2) if there is circulation,

$$\delta \int L\,dt = \left[\rho\Sigma \int \kappa\,\frac{d\,\delta\phi}{dn}\,d\sigma\right]$$
$$= \left[-\rho\Sigma \int \phi_1\,\frac{d\,\delta\phi}{dn}\,dS\right],$$

by § 10; hence, if ϕ_1, the velocity potential of the circulation, is supposed unvaried, and therefore κ_1, κ_2 ... constants, we have

$$\delta \int L'\,dt = 0,$$

where

$$L' = L + \rho\Sigma \int \phi_1\,\frac{d\phi}{dn}\,dS,$$

for given initial and terminal positions of the solids, given circulations, and given time of motion.—*Sept.* 4.]

Thursday, April 3rd, 1884.

Professor HENRICI, F.R.S., President, in the Chair.

The Rev. A. C. E. Blomfield was admitted into the Society.

The following communications were made :—

"On Double Algebra," Professor Cayley, F.R.S.

"On direct investigation of the Complete Primitive of the Equation $F(x, y, z, p, q) = 0$, with a way of remembering the Auxiliary System," Mr. J. W. Russell.

"On Electrical Oscillations and the effects produced by the motion of an Electrified Sphere," J. J. Thomson, F.R.S.

"On the Homogeneous and other forms of Equation of a Plane Section of a Surface," J. J. Walker, F.R.S.

The following presents were received :—

Cabinet Portraits of Professor Cayley and the late Professor J. Clerk Maxwell, presented by Mr. R. Forsyth Scott.

"American Journal of Mathematics," Vol. vi., No. 3.

"Educational Times," April, 1884.

On Double Algebra. *By* Professor Cayley.

[*Read April 3rd,* 1884.]

1. I consider the Double Algebra formed with the extraordinary symbols, or "extraordinaries" x, y, which are such that

$$x^2 = ax + by,$$
$$xy = cx + dy,$$
$$yx = ex + fy,$$
$$y^2 = gx + hy,$$

or, as these equations may also be written,

	x	y
x	(a, b)	(c, d)
y	(e, f)	(g, h)

where a, b, c, d, e, f, g, h are ordinary symbols, or say coefficients; all coefficients being commutative and associative *inter se*, and with the extraordinaries x, y.

The system depends in the first instance on the eight parameters a, b, c, d, e, f, g, h; but we may, instead of the extraordinaries x, y, consider the new extraordinaries connected therewith by the linear relations $\xi = \alpha x + \beta y$, $\eta = \gamma x + \delta y$, where the coefficients $\alpha, \beta, \gamma, \delta$ may be determined so as to establish between the eight parameters any four relations at pleasure (or, what is the same thing, $\alpha, \beta, \gamma, \delta$ are what I call "apoclastic" constants) : and the number of parameters is thus properly $8-4, = 4$.

2. The extraordinaries here considered are not in general associative; differing herein from the imaginaries of Peirce's Memoir, "Linear Associative Algebra" (1870), reprinted in the *American Mathematical Journal*, t. 4 (1881), pp. 97—227, which, as appears by the title, refers only to associative imaginaries. I recall some definitions and results. The symbol x is said to be *idempotent* if $x^2=x$, *nilpotent* if $x^2=0$; and the systems of associative symbols are expressed as much as may be by means of such idempotent and nilpotent symbols: thus the linear systems are $(a_1)\ x^2 = x$, $(b_1)\ x^2 = 0$. A double system composed of independent symbols, that is, symbols x, y each belonging to its own linear system, and moreover such that $xy = yx = 0$, is said to be "mixed"; thus the mixed double systems are

	x	y
x	x	0
y	0	y

,

	x	y
x	x	0
y	0	0

,

	x	y
x	0	0
y	0	0

.

But these Peirce excludes from consideration, attending only to the pure systems, which he finds to be

(a_2)		x	y
	x	x	y
	y	y	0

,

(b_2)		x	y
	x	x	y
	y	0	0

,

(c_2)		x	y
	x	y	0
	y	0	0

.

To these, however, should be added the system

(d_2)		x	y
	x	x	0
	y	y	0

;

see *post*, No. 19.

3. In the general theory, where the symbols are not in the first instance taken to be associative, we may of course establish between the coefficients such relations as will make the symbols associative, and the question presents itself to show how in this case the system reduces itself to one of Peirce's systems. This I considered in my note "On Associative Imaginaries," *Johns Hopkins University Circular*, No. 15 (1882), p. 211; I there obtained as the general form of the commutative and associative system

$$x^2 = ax + by,$$
$$xy = yx = cx + dy,$$
$$y^2 = \frac{cd}{b}x + \frac{d^2+bc-ad}{b}y,$$

the relation of which to Peirce's system was, as I there remarked, pointed out to me by Mr. C. S. Peirce: this will be considered in the sequel, Nos. 13 to 19.

4. Starting now with the general equations

$$x^2 = ax+by,$$
$$xy = cx+dy,$$
$$yx = ex+fy,$$
$$y^2 = gx+hy,$$

we may attempt to find an extraordinary $\xi, = \alpha x+\beta y$ (α, β coefficients) such that $\xi^2 = K\xi$ (K a coefficient). K is not in general $= 0$, and, when it is not $= 0$, it may without loss of generality be taken to be $= 1$; we have then $\xi^2 = \xi$, ξ an *idempotent* symbol. But K may be $= 0$; and then $\xi^2 = 0$, ξ a *nilpotent* symbol. To include the two cases, I retain K, it being understood that, when K is not $= 0$, it may be taken to be $= 1$.

We have $\xi^2 = \alpha^2(ax+by)+\alpha\beta(\overline{c+e}\,x+\overline{d+f}\,y)+\beta^2(gx+hy)$
$= \{a\alpha^2+(c+e)\,\alpha\beta+g\beta^2\}x+\{b\alpha^2+(d+f)\,\alpha\beta+h\beta^2\}y.$

Hence, when this is $= K\xi$, that is,
$= K(\alpha x+\beta y)$,

we have
$$\frac{\alpha}{\beta} = \frac{a\alpha^2+(c+e)\,\alpha\beta+g\beta^2}{\alpha^2+(d+f)\,\alpha\beta+h\beta^2},$$

a cubic equation for the determination of the ratio $\alpha : \beta$; and, for any particular value of the ratio, we can in general determine the absolute magnitudes, so that

$$K, = \frac{1}{\alpha}\{a\alpha^2+(c+e)\,\alpha\beta+g\beta^2\}, = \frac{1}{\beta}\{b\alpha^2+(d+f)\,\alpha\beta+h\beta^2\},$$

shall be $=1$. If, however, for the given value of the ratio we have

$$a\alpha^2+(c+e)\,\alpha\beta+g\beta^2 = 0, \quad b\alpha^2+(d+f)\,\alpha\beta+h\beta^2 = 0$$

(one of these equations, of course, implying the other), then the value of K is $= 0$.

5. It follows that there are in general three idempotent symbols ξ, η, ζ, that is, extraordinaries such that $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$. The cubic equation may, however, have two equal roots, or three equal roots, or it may vanish identically; in this last case, any linear function $\alpha x+\beta y$ is in general idempotent. But (as will be considered in detail further on) we may, instead of an idempotent symbol or symbols, have a nilpotent symbol or symbols. It might be convenient to use the term Potency for a symbol which is in general idempotent, but which may be nilpotent. Writing $\frac{\alpha}{\beta} = \frac{-y}{x}$, we obtain a cubic

equation $\Omega = (x, y)^3 = 0$, where obviously the linear factors of Ω are the just-mentioned functions ξ, η, ζ; that is, we have ξ, η, ζ as the linear factors of the cubic function

$$\Omega, = gx^3 + (h-c-e)\,x^2y + (a-d-f)\,xy^2 + by^3;$$

each such factor, except in the case where it is nilpotent, being determined so that it shall be idempotent. The cubic function of course vanishes identically if $g = 0$, $h-c-e = 0$, $a-d-f = 0$, $b = 0$.

6. Two extraordinaries $\xi, = \alpha x + \beta y$; $\eta, = \gamma x + \delta y$ ($\alpha, \beta, \gamma, \delta$ coefficients) may be such that $\xi\eta = 0$ (this, of course, does not imply $\eta\xi = 0$): we have in fact

$$\begin{aligned}\xi\eta &= (\alpha x + \beta y)(\gamma x + \delta y)\\ &= \alpha\gamma x^2 + \alpha\delta xy + \beta\gamma yx + \beta\delta y^2\\ &= \alpha\gamma\,(ax+by) + \alpha\delta\,(cx+dy) + \beta\gamma\,(ex+fy) + \beta\delta\,(gx+hy)\\ &= (a\alpha\gamma + c\alpha\delta + e\beta\gamma + g\beta\delta)\,x + (b\alpha\gamma + d\alpha\delta + f\beta\gamma + h\beta\delta)\,y;\end{aligned}$$

and the required condition is satisfied if

$$\begin{aligned}a\alpha\gamma + c\alpha\delta + e\beta\gamma + g\beta\delta &= 0,\\ b\alpha\gamma + d\alpha\delta + f\beta\gamma + h\beta\delta &= 0.\end{aligned}$$

Writing these equations first under the form

$$\begin{aligned}\gamma\,(a\alpha + e\beta) + \delta\,(c\alpha + g\beta) &= 0,\\ \gamma\,(b\alpha + f\beta) + \delta\,(d\alpha + h\beta) &= 0,\end{aligned}$$

and then under the form

$$\begin{aligned}\alpha\,(a\gamma + c\delta) + \beta\,(e\gamma + g\delta) &= 0,\\ \alpha\,(b\gamma + d\delta) + \beta\,(f\gamma + h\delta) &= 0,\end{aligned}$$

we have $(a\alpha + e\beta)(d\alpha + h\beta) - (b\alpha + f\beta)(c\alpha + g\beta) = 0$,

a quadric equation for the determination of $\alpha : \beta$; and then

$$(a\gamma + c\delta)(f\gamma + h\delta) - (b\gamma + d\delta)(e\gamma + g\delta) = 0,$$

a quadric equation of the determination of $\gamma : \delta$; that is, there are two values of the left-hand factor ξ; and two values of the right-hand factor η. But, of course, these correspond each to each, viz., either factor being given, the other factor is determined uniquely.

Writing successively $\frac{\alpha}{\beta} = \frac{-y}{x}$, and $\frac{\gamma}{\delta} = \frac{-y}{x}$, we have the quadric functions $(x, y)^2$,

$$\begin{aligned}\Phi &= (eh-fg)\,x^2 + (-ah-de+bg+cf)\,xy + (ad-bc)\,y^2,\\ \Phi_1, &= (ch-dg)\,x^2 + (-ah-cf+bg+de)\,xy + (af-be)\,y^2,\end{aligned}$$

where the linear factors of Φ are the two values ξ_1, ξ_2 of the left-hand factor ξ; and the linear factors of Φ_1 are the two values η_1, η_2 of the right-hand factor η.

7. In the commutative case, $c = e$, and $d = f$, we have

$$\Phi = \Phi_1 = (ch - dg)\, x^2 + (-ah + bg)\, xy + (ad - bc)\, y^2;$$

here $\xi\eta = 0$, $\eta\xi = 0$, and the values may be taken to be $(\xi_1, \eta_1) = (\xi, \eta)$, $(\xi_2, \eta_2) = (\eta, \xi)$, so that $\xi_1\xi_2 = \xi\eta$, $= \eta_1\eta_2$; that is, $\Phi = \Phi_1$, as above. The value of Ω is $\Omega = gx^3 + (h - 2c)\, x^2y + (a - 2d)\, xy^2 + by^3$.

8. In the commutative and associative case, taking ξ, η, ζ to be the three idempotent symbols, $\xi^2 = \xi$, $\eta^2 = \eta$, $\zeta^2 = \zeta$, we have

$$\xi(\eta - \xi\eta), = \xi\eta - \xi^2\eta, = \xi\eta - \xi\eta = 0;$$

and in this manner we have the six equations

$$\xi(\eta - \xi\eta) = 0,\ \xi(\zeta - \xi\zeta) = 0;\quad \eta(\xi - \eta\xi) = 0,\ \eta(\zeta - \eta\zeta) = 0;$$

$$\zeta(\xi - \zeta\xi) = 0,\ \zeta(\eta - \zeta\eta) = 0;$$

viz., regarding the right-hand factor as being in each case expressed as a linear function of x, y, we have apparently six products of two linear factors, each $= 0$; there is only one such product $\Phi = 0$, hence, disregarding coefficients, each of the six products must be $= \Phi$, or it must be identically $= 0$, viz., this will be the case if the second factor be $= 0$. We hence conclude that two of the symbols ξ, η, ζ, suppose ξ and η, must be factors of Φ, viz., Φ must be $= \xi\eta$. We have $\Omega = \xi\eta\zeta$; consequently $\Omega = \zeta\Phi$, that is, two of the three linear factors of Ω are the symbols ξ, η, which are such that $\xi\eta\ (= \eta\xi) = 0$. To complete the theory, observe that ζ must be a linear function of ξ, η, $= a\xi + b\eta$ suppose (a, b coefficients, neither of them $= 0$); we thence have

$$\zeta^2, = a^2\xi^2 + b^2\eta^2, = a^2\xi + b^2\eta, = \zeta, = a\xi + b\eta;$$

that is, $(a^2 - a)\, \xi + (b^2 - b)\, \eta = 0$; whence $a = 1$, $b = 1$, and therefore $\zeta = \xi + \eta$; hence also $\xi\zeta = \xi$ and $\eta\zeta = \eta$; $\zeta - \xi\zeta = \eta$, $\zeta - \eta\zeta = \xi$. The six products consequently are $\xi\eta$, $\xi\eta$, $\eta\xi$, $\eta\xi$, $\zeta 0$, $\zeta 0$, each $= \Phi$ or identically $= 0$.

9. In verification of the theorem that for the commutative and associative system the cubic function Ω contains the quadric function ϕ as a factor, we may write, as above,

$$g = \frac{cd}{b},\quad h = \frac{d^2 + bc - ad}{b},$$

values which give

$$b\Phi = (bc^2 - acd)\, x^2 + (-ad^2 - abc + a^2d + bcd)\, xy + (abd - b^2c)\, y^2$$
$$= -(ad - bc)\, \{cx^2 + (d - a)\, xy - by^2\},$$

$$b\Omega = cdx^3 + (d^2 - bc - ad)\, x^2y + (ab - 2bd)\, xy^2 + b^2y^3,$$
$$= (dx - by)\, \{cx^2 + (d - a)\, xy - by^2\},$$

which gives the theorem in question. And observe further that

$$(dx-by)^2 = d^2(ax+by)-2bd(cx+dy)+b^2\left(\frac{cd}{b}x+\frac{d^2+bc-ad}{b}y\right),$$
$$= (ad-bc)(dx-by).$$

That is, disregarding coefficients, the two idempotent symbols ξ, η are the linear factors of $cx^2+(d-a)xy-by^2$, and the third idempotent symbol ζ is $= dx-by$.

10. Introducing coefficients in order to make the symbols ξ, η, ζ idempotent, and writing accordingly

$$\xi = \frac{1}{K}\{cx+\tfrac{1}{2}(d-a+\surd\nabla)y\}, \quad \nabla = (d-a)^2+4bc, \text{ so that}$$

$$\eta = \frac{1}{L}\{cx+\tfrac{1}{2}(d-a-\surd\nabla)y\}, \qquad \xi\eta = \frac{c}{KL}\{cx^2+(d-a)xy-by^2\},$$

$$\zeta = \frac{1}{P}(dx-by),$$

we have to verify that it is possible to determine K, L, P so that $\xi^2=\xi$, $\eta^2=\eta$, $\zeta^2=\zeta$, $\zeta=\xi+\eta$. The last equation gives

$$\frac{d}{P} = \frac{c}{K} + \frac{c}{L},$$

$$-\frac{2b}{P} = \frac{d-a+\surd\nabla}{K}+\frac{d-a-\surd\nabla}{L},$$

and we thence have

$$\frac{d(d-a)+2bc-d\surd\nabla}{P} = -\frac{2c\surd\nabla}{K},$$

$$\frac{d(d-a)+2bc+d\surd\nabla}{P} = \frac{2c\surd\nabla}{L},$$

and we can from the equation $\zeta^2=\zeta$ find P; viz., comparing the coefficients of x, we have

$$\frac{d}{P} = \frac{1}{P^2}\left(d^2a-2bdc+b^2\frac{cd}{b}\right), = \frac{d}{P^2}(ad-bc), \text{ that is, } P = (ad-bc),$$

or the values of K and L are

$$\frac{\{d(d-a)+2bc-d\surd\nabla\}}{ad-bc} = -\frac{2c\surd\nabla}{K},$$

$$\frac{\{d(d-a)+2bc+d\surd\nabla\}}{ad-bc} = +\frac{2c\surd\nabla}{L},$$

which should agree with the values of K and L found from the equations $\xi^2=\xi$, $\eta^2=\eta$, respectively. Comparing the coefficients of x,

the first of these equations gives

$$\frac{c}{K} = \frac{1}{K^2}\left\{c^2a + c(d-a+\sqrt{\nabla})c + \tfrac{1}{4}(d-a+\sqrt{\nabla})^2\frac{cd}{b}\right\}$$

$$= \frac{c}{4bK^2}\left\{\begin{array}{l} 4abc + 4bc(d-a) + d\{(d-a)^2+\nabla\} \\ \qquad\qquad + 2\{2bc + d(d-a)\}\sqrt{\nabla} \end{array}\right\}$$

$$= \frac{c}{2bK^2}\left\{d\nabla + \{d(d-a)+2bc\}\sqrt{\nabla},\right\}$$

that is,
$$K = \frac{1}{2b}\sqrt{\nabla}\{d(d-a+2bc)+d\sqrt{\nabla}\},$$

and the equation for K becomes

$$\frac{d(d-a)+2bc-d\sqrt{\nabla}}{ad-bc} = \frac{-4bc}{d(d-a)+2bc+d\sqrt{\nabla}};$$

that is,
$$\{d(d-a)+2bc\}^2 - d^2\{(d-a)^2+4bc\} = -4bc(ad-bc),$$

which is right; and similarly the equation for L leads to this same equation.

11. We may now establish, on the principles appearing in No. 5, the different forms of the system. Using *idem* and *nil* as abbreviations for idempotent and nilpotent respectively, there are in all 11 cases.

(1) 3 idems. Taking two of these to be x and y, the system is

$$x^2 = x, \quad xy = cx+dy, \quad yx = ex+fy, \quad y^2 = y.$$

Hence $\Omega = (1-c-e)\,x^2y + (1-d-f)\,xy^2$, so that the third factor is $(1-c-e)\,x + (1-d-f)\,y$. This must not reduce itself to x or y, for, if so, there would be a twofold idem; viz., as negative conditions we must have $c+e \neq 1$, $d+f \neq 1$.

And we have

$$\{(1-c-e)x + (1-d-f)y\}^2$$
$$= \{1-(c+e)(d+f)\}\,[(1-c-e)x + (1-d-f)y],$$

which must be an idem: viz., we have the further negative condition $(c+e)(d+f) \neq 1$.

(2) 2 idems and 1 nil. This arises from (1) by assuming therein

$$(c+e)(d+f) = 1, \text{ say } d+f = \frac{1}{c+e}:$$

for then, writing $z = -(c+e)x+y$, we have

$$z^2 = (c+e)^2x - (c+e)\{(c+e)x + (d+f)y\} + y,$$
$$= \{1-(c+e)(d+f)\}\,y, = 0;$$

viz., z is a nil. And, if in the equations instead of the idem y we introduce the nil z, then the equations assume the form

$x^2 = x$, $xz = [(c+e)d-e]x + dz$, $zx = [(c+e)f-c]x + fz$, $z^2 = 0$; with the idem $y = (c+e)x + z$: hence the negative conditions $c+e \neq 1$ or 0, implying $d+f \neq 1$.

But the equations are obtained in a more simple form by taking x for the idem and y for the nil; viz., we then have $x^2 = x$, $xy = cx+dy$, $yx = ex+fy$, $y^2 = 0$: we must then have $z, = -(c+e)x+(1-d-f)y$ for an idem; this gives $z^2 = -(c+e)(d+f)z$, and we have the negative conditions $c+e \neq 0$, $d+f \neq 0$ or 1.

(3) 1 idem and 2 nils. This may be deduced from (2) by writing therein $d+f = 0$; for then $z, = -(c+e)x+y$ is a nil. The equations are $x^2 = x$, $xy = cx+dy$, $yx = ex-dy$, $y^2 = 0$: and if, instead of x, we introduce therein z by the equation $z = -(c+e)x+y$, the equations become $y^2 = 0$, $yz = [-e+d(c+e)]y-ez$, $zy = [-c-d(c+e)]y+cz$, $z^2 = 0$, with the negative condition $c+e \neq 0$.

But it is more simple to take x, y as the nils: the equations then are $x^2 = 0$, $xy = cx+dy$, $yx = ex+fy$, $y^2 = 0$. We must have $z, = (c+e)x+(d+f)y$, an idem: this gives $z^2 = (c+e)(d+f)z$; and we have the negative conditions $c+e \neq 0$, $d+f \neq 0$.

(*) We cannot have three nils. For in (3) to make z a nil we must have $c+e = 0$ or $d+f = 0$, and in the two cases respectively $z, = (c+e)x+(d+f)y$ becomes $= x$ and $= y$; so that x or y is a twofold nil. Or, what comes to the same thing, we have

$$\Omega = -(c+e)x^2y-(d+f)xy^2,$$

and Ω has a twofold factor if $c+e = 0$ or $d+f = 0$.

(4) A twofold idem and a onefold idem. Taking x for the twofold idem and y for the onefold idem, Ω must reduce itself to $(1-c-e)x^2y$, viz., we must have $d+f = 1$, or say $f = 1-d$. The equations are $x^2 = x$, $xy = cx+dy$, $yx = ex+(1-d)y$, $y^2 = y$; and we have the negative condition $c+e \neq 1$, for otherwise Ω would vanish identically.

(5) A twofold idem and a onefold nil. Taking x for the twofold idem and y for the onefold nil, then the equations are $x^2 = x$, $xy = cx+dy$, $yx = ex+(1-d)y$, $y^2 = 0$; and we have the negative condition $c+e \neq 0$.

(6) A twofold nil and a onefold idem. Taking these to be x and y, then $d+f = 0$, and the equations are $x^2 = 0$, $xy = cx+dy$, $yx = ex-dy$, $y^2 = y$; and we have the negative condition $c+e \neq 1$.

(7) A twofold nil and a onefold nil. Taking these to be x and y, we have $d+f = 0$, and the equations are $x^2 = 0$, $xy = cx+dy$, $yx = ex-dy$, $y^2 = 0$; with the negative condition $c+e \neq 0$.

(8) A threefold idem. Taking this to be x, then Ω must reduce itself to gx^3, viz., we must have $h = c+e$, $1 = d+f$; and the equations are $x^2 = x$, $xy = cx+dy$, $yx = ex+(1-d)y$, $y^2 = gx+(c+e)y$; we have the negative condition $g \neq 0$, for otherwise Ω would vanish identically.

(9) A threefold nil. Taking this to be x, then we must have $h = c+e$, $0 = d+f$; the equations are $x^2 = 0$, $xy = cx+dy$, $yx = ex-dy$, $y^2 = gx+(c+e)\,y$; and there is again the negative condition $g \neq 0$.

(10) $\Omega = 0$ identically: infinity of idems, 1 nil. Ω will vanish identically if $g = 0$, $h = c+e$, $a = d+f$, $b = 0$. If there is 1 idem, there will be an infinity of idems, and 1 nil. For, assume an idem x, $x^2 = x$; and, if possible, let there be no other idem; then there will be a nil y, $y^2 = 0$. We have $c+e = 0$, $d+f = 1$; and the equations are $x^2 = x$, $xy = cx+dy$, $yx = -cx+(1-d)\,y$, $y^2 = 0$; whence $xy+yx = y$. Taking α, β arbitrary coefficients, we have

$$(\alpha x+\beta y)^2 = \alpha^2 x+\alpha\beta y, = \alpha\,(\alpha x+\beta y);$$

hence $\alpha x+\beta y$ is an idem, except in the case $\alpha = 0$, when it is the original nil y.

If besides the idem x we have an idem y, then the conditions are $c+e = 1$, $d+f = 1$: the equations are

$$x^2 = x,\ xy = cx+dy,\ yx = (1-c)\,x+(1-d)\,y,\ y^2 = y;$$

whence $xy+yx = x+y$. Considering the combination $\alpha x+\beta y$, we have

$$(\alpha x+\beta y)^2 = \alpha^2 x+\alpha\beta\,(x+y)+\beta^2 y, = (\alpha+\beta)(\alpha x+\beta y).$$

This is an idem, except in the case $\alpha+\beta = 0$, when it is a nil; or, say we have the single nil $x-y$. We have thus again an infinity of idems, 1 nil.

(11) $\Omega = 0$ identically; an infinity of nils. Taking the two nils x and y, the conditions are $c+e = 0$, $d+f = 0$; the equations are $x^2 = 0$, $xy = cx+dy$, $yx = -cx-dy$, $y^2 = 0$; whence $xy+yx = 0$. Considering the arbitrary combination $\alpha x+\beta y$, we have

$$(\alpha x+\beta y)^2 = \alpha\beta\,(xy+yx), = 0,$$

viz., $\alpha x+\beta y$ is a nil; or there are an infinity of nils.

12. The different cases may be grouped together as follows:—

A. 2 idems, (1), (2), (4), (10).
Equations $x^2 = x$, $xy = cx+dy$, $yx = ex+fy$, $y^2 = y$.

B. 1 idem and 1 nil, (2), (3), (5), (6), (10).
Equations $x^2 = x$, $xy = cx+dy$, $yx = ex+fy$, $y^2 = 0$.

C. 2 nils, (3), (7), (11).
Equations $x^2 = 0$, $xy = cx+dy$, $yx = ex+fy$, $y^2 = 0$.

D. Threefold idem, (8).
Equations $x^2 = x$, $xy = cx+dy$, $yx = ex-(1-d)\,y$,
$y^2 = gx+(c+e)\,y$.

E. Threefold nil, (9).

Equations $x^2 = 0,\ xy = cx + dy,\ yx = ex - dy,$
$y^2 = gx + (c+e)\,y.$

The several cases of A, B, C respectively are distinguished by negative conditions which need not be here repeated.

13. I consider, as in my Note before referred to, the conditions in order that the system may be associative. We have the 8 products, x^3, x^2y, xyx, xy^2, yx^2, y^2x, yxy, y^3, giving rise to equations $x \,.\, x^2 = x^2 \,.\, x$, $x \,.\, xy = x^2 \,.\, y$, ..., $y \,.\, y^2 = y^2 \,.\, y$, which, on putting therein for x^2, xy, yx, y^2 their values, must be satisfied identically. We thus obtain in the first instance 16 relations, but some of these are repeated, and we have actually only 12 relations; viz., the relations are

(twice) $b\,(c-e) = 0,$

$b\,(f-d) = 0,$

$g\,(c-e) = 0,$

(twice) $g\,(f-d) = 0,$

(twice) $bg - cd = 0,$

(twice) $bg - ef = 0,$

$c\,(c-h) + g\,(d-a) = 0,$

$d\,(d-a) + b\,(c-h) = 0,$

$e\,(e-h) + g\,(f-a) = 0,$

$b\,(e-h) + f\,(f-a) = 0,$

$a\,(c-e) - cf + de = 0,$

$h\,(f-d) - cf + de = 0.$

14. From the first four equations it appears that either $b = 0$, $g = 0$, or else $c = e$ and $d = f$. I attend first to the latter case, viz., we have here the commutative system

$$x^2 = ax + by,\ \ xy = yx = cx + dy,\ \ y^2 = gx + hy.$$

In order that this may be associative, we must still have the relations

$$bg - cd = 0,$$
$$c\,(c-h) + g\,(d-a) = 0,$$
$$d\,(d-a) + b\,(c-h) = 0,$$

or, as they may be written

$$\begin{vmatrix} b, & -c, & d-a \\ -d, & g, & c-h \end{vmatrix} = 0.$$

These are satisfied by $g = \frac{cd}{b}$, $h = \frac{d^2+bc-ad}{b}$, and we have thus the commutative and associative system of the Note.

Every system is of the form A, B, C, D, or E; and it can be shown that the commutative and associative system is not of the form D. For, if D were commutative, we should have $e = c$, $d = \frac{1}{2}$, viz., the equations will be $x^2 = x$, $xy = yx = cx + \frac{1}{2}y$, $y^2 = gx + 2cy$, that is,

$$a, b, c, d, g, h = 1, 0, c, \tfrac{1}{2}, g, 2c;$$

and the last of the three relations, viz., $d(d-a)+b(c-h) = 0$, would thus be $\frac{1}{2}(\frac{1}{2}-1) = 0$, which is not satisfied. Hence the commutative and associative system can only be of one of the forms A, B, C, and E.

15. First, if the form be A, B, or C; there will be the two idem-or-nil symbols x and y, that is, we may assume $b = 0$, $g = 0$; and the associative conditions then become $cd = 0$, $c(c-h) = 0$, $d(d-a) = 0$, viz., for the forms A, B, C,

A. $x^2 = x$, $y^2 = y$,

B. $x^2 = x$, $y^2 = 0$,

C. $x^2 = 0$, $y^2 = 0$,

these are $cd = 0$, $c(c-1) = 0$, $d(d-1) = 0$; $c = 0$ or 1, $d = 0$ or 1,

„ $cd = 0$, $c^2 = 0$, $d(d-1) = 0$; $c = 0$, $d = 0$ or 1,

„ $cd = 0$, $c^2 = 0$, $d^2 = 0$; $c = 0$, $d = 0$.

But for the form A., if $c=0$, $d=1$, that is, $xy = yx = y$, then, writing $z = x-y$, we have $z^2 = z$, $yz = zy = 0$, $y^2 = y$. And similarly, if $c = 1$, $d = 0$, that is, $xy = yx = x$, then, writing $z = -x+y$, we have $z^2 = z$, $zx = xz = 0$, $x^2 = x$. That is, each of these is reduced to the first case $c = 0$, $d = 0$; that is, $x^2 = x$, $xy = yx = 0$, $y^2 = y$.

For the form B., if $c = 0$, $d = 1$, then the system is $x^2 = x$, $xy = yx = y$, $y^2 = 0$; and this cannot be reduced to the first case $x^2 = x$, $xy = yx = 0$, $y^2 = 0$.

For the form C., there is only one case, as above.

For the form E., we have $a = 0$, $b = 0$, ($c = e$, $d = 0$, in order that the system may be commutative), $h = 2c$, viz., the equations must be $x^2 = 0$, $xy = yx = cx$, $y^2 = gx + 2cy$. The associative conditions then give $c = 0$; or, the system is $x^2 = 0$, $xy = yx = 0$, $y^2 = gx$. Writing $\frac{x}{g}$ instead of x; and for convenience interchanging x and y, the equations are $x^2 = y$, $xy = yx = 0$, $y^2 = 0$.

16. The commutative associative system is thus seen to be reducible

as follows :—

A. system is $x^2 = x$, $xy = yx = 0$, $y^2 = y$, first mixed system see No. 2.

B. „ $x^2 = x$, $xy = yx = y$, $y^2 = 0$, Peirce's system (a_2).

Or else

B. „ $x^2 = x$, $xy = yx = 0$, $y^2 = 0$, second mixed system.

C. „ $x^2 = 0$, $xy = yx = 0$, $y^2 = 0$, third mixed system.

E. „ $x^2 = y$, $xy = yx = 0$, $y^2 = 0$, Peirce's system (c_2).

I said, at the end of my Note before referred to, that it had been pointed out to me "that my system [the commutative associative system], in the general case $ad-bc$ not $=0$, is expressible as a mixture of two algebras of the form (a_1), see p. 120; whereas, if $ad-bc=0$, it is reducible to the form (c_2), see p. 122." The accurate conclusion is as above, that the commutative associative system is either a mixed system of one of the three forms, or else a system (a_2), or (c_2).

17. Considering next the non-commutative associative systems, we have here, *ante*, No. 14, $b=0$, $g=0$; and the relations which remain to be satisfied then are

$$cd=0,\ ef=0,\ c(c-h)=0,\ d(d-a)=0,\ e(e-h)=0,\ f(f-a)=0,$$
$$a(c-e)-cf+de=0,\quad h(f-d)-cf+de=0.$$

The first equation gives $cd=0$, that is, $c=0$ or $d=0$; but we may attend exclusively to the case $c=0$, for the case $d=0$ may be deduced from this by the interchange of x, y. We have then $ef=0$; and it will be convenient to separate the cases

I. $c=0, e=0, f=0$, giving $d(d-a)=0$, $dh=0$,

II. $c=0, e=0, f\neq 0$, „ $d(d-a)=0, f-a=0, h(f-d)=0$,

III. $c=0, e\neq 0, f=0$, „ $d(d-a)=0, e-h=0, (d-a)=0, d(e-h)=0$,

that is, $d-a=0$, $e-h=0$.

18. We have thus five cases

I. (a). $d=0$: $x^2=ax$, $xy=yx=0$, $y^2=hy$: commutative, and so included in what precedes.

I. (b). $d=a$, $h=0$: $x^2=ax$, $xy=ay$, $yx=0$, $y^2=0$: or, writing as we may do $a=1$, this is $x^2=x$, $xy=y$, $yx=0$, $y^2=0$; which is Peirce's system (b_2).

II. (c). $d=f=a$: $x^2=ax$, $xy=yx=ay$, $y^2=hy$: commutative, and so included in what precedes.

II. (d). $d=0$, $f=a$, $h=0$: $x^2=ax$, $xy=0$, $yx=ay$, $y^2=0$; or, writing as we may do $a=1$, this is $x^2=x$, $xy=0$, $yx=y$, $y^2=0$; which is the system (d_2).

III. (*e*). $d = a$, $e = h$: $x^2 = ax$, $xy = ay$, $yx = hx$, $y^2 = hy$; or, writing as we may do $a=1$, $h=1$, this is $x^2=x$, $xy=y$; $yx = x$; $y^2 = y$. Introducing here the new symbol z, $= x-y$, we have $z^2 = 0$, $xz = z$, $zx = 0$, $yz = z$, $zy = 0$. Thus x, z form the system $x^2 = x$, $xz = z$, $zx = 0$, $z^2 = 0$ (or, what is the same thing, y, z form a system $y^2 = y$, $yz = z$, $zy = 0$, $z^2 = 0$) ; each of these is Peirce's system (b_2).

The conclusion is that every non-commutative associative system is either Peirce's system (b_2), or else the omitted system (d_2). Hence, disregarding the mixed systems, every associative system is either (a_2), (b_2), (c_2), or (d_2).

19. It may be proper to show that the systems (b_2), $x^2 = x$, $xy = y$, $yx = 0$, $y^2 = 0$, and (d_2), $x^2 = x$, $xy = 0$, $yx = y$, $y^2 = 0$, or say

	a	b	c	d	e	f	g	h
(b_2)	1	0	0	1	0	0	0	0
(d_2)	1	0	0	0	0	1	0	0,

are really distinct from each other. Observe that they each belong to the case 10, $\Omega = 0$, an infinity of idems and 1 nil; viz., in each of them writing $z = x+\beta y$, β an arbitrary coefficient, we have $z^2 = x^2+\beta(xy+yx)+\beta^2y^2$, $= x+\beta y$, $= z$, we have z an idem, and y is the only nil. And, this being so, we have in the first system $zy = y$, $yz = 0$, viz., the system is $z^2 = z$, $zy = y$, $yz = 0$, $y^2 = 0$, retaining, when we write z for x, its original form. And similarly, in the second system, $zy = 0$, $yz = y$; viz., the system is $z^2 = z$, $zy = 0$, $yz = y$, $y^2 = 0$, retaining, when we write therein z for x, its original form. The two are thus distinct systems, in no wise transformable the one into the other.

On Electrical Oscillations and the effects produced by the motion of an Electrified Sphere. *By* J. J. Thomson, M.A., Fellow and Assistant Lecturer of Trinity College, Cambridge.

[*Read April 3rd*, 1884.]

In this paper two problems are discussed which, though physically different, are yet capable of solution by almost the same mathematical treatment.

The first problem treats of the vibrations which take place in the electrical distribution on the surface of a thin spherical shell when the

cause which produced an irregular distribution over the surface is suddenly withdrawn. The electrical oscillations, which occur when the oppositely charged plates of a condenser are connected, are due to the interchange of energy between the kinetic energy possessed by the currents in the wire joining the plates of the condenser and the potential energy of the electricity distributed over these plates. In the problem we are considering, the oscillations are due to the interchange of energy between the currents which start off to equalize the distribution of electricity and the potential energy of this distribution. It is necessary to distinguish between two cases. The first is when the electrical system which produced the irregular distribution over the shell is inside it. In this case, if the shell be made of a perfectly conducting material, there will be neither magnetic nor electro-motive forces outside it; thus all the energy will be kept inside the shell, so that, were it not for the resistance of the shell, the vibrations would last for ever. In the second case, the electrical system which produces the distribution is outside the shell. In this case there will be neither magnetic nor electro-motive forces inside the shell, but a magnetic field will be produced outside the shell, and energy will continually be propagated from the shell into the surrounding dielectric. It is proved in the following paper, that the dissipation of the energy of the electrical distribution due to this cause is so great that the whole of the energy will be dissipated in the course of a few oscillations.

he period of the oscillations in both cases is investigated, and it is proved that the time of oscillation is comparable with the time taken by light to cross over the diameter of the spherical shell. In particular, if a distribution of electricity over the shell represented by the first zonal harmonic be produced by an internal electrical system, the period of vibration equals the time taken by light to pass over a distance of 1·14 times the diameter of the shell. In this case the magnetic forces inside the shell might be made sufficiently large to be easily measurable, so that this result seems to afford a promising way of testing Maxwell's theory experimentally. The very rapid dissipation of the energy by dissipation into the surrounding medium forbids us to expect that a collection of small particles could be made luminous by electrical means; the vibrations would not last long enough for us to see them.

It would require a very sudden disturbance to produce these oscillations, as their period is very small. Supposing, for example, the disturbance to be produced by the motion of an electrified point, to give rise to these oscillations it would have to move with a velocity comparable with that of light. We might get a sudden disturbance, however, by increasing the charge on a conductor until the potential became so great that a spark passed; the time occupied in

the discharge would be small, and we might expect the electricity on the neighbouring conductors to vibrate.

We might get, by means of suitable electrical systems, vibrations of all periods, from those of Light to those of Sound.

The case of a solid sphere is not given in this paper, because the energy due to any distribution of electricity on its surface is very rapidly dissipated into the surrounding dielectric, just as in the case of a spherical shell excited by an external electrical system.

In the second part of this paper, the effects produced by the motion of an originally uniformly electrified sphere are discussed. I considered this problem in a paper published in the *Philosophical Magazine*, for April 1881, neglecting the self-induction of the displacement currents; in this paper, however, these are taken into account, and the sphere is supposed to be of finite size. The most interesting results of this investigation are, that, whether the velocity of the sphere be uniform or variable, the charged sphere produces the same magnetic force as unit length of a current whose intensity is ve and which flows in the direction of motion of the sphere,—v being the velocity of, and e the charge on, the sphere. If the sphere is vibrating in a magnetic field perpendicularly to the direction of the magnetic induction, the force on the sphere is perpendicular both to its direction of motion and the magnetic induction, and

$$= evb \left\{ \frac{2}{3} \frac{\mu}{\mu'} - 1 \right\},$$

where b is the magnetic induction, and μ and μ' the coefficients of magnetic permeability of the dielectric and the sphere respectively. It is also proved that the distribution of electricity on the surface of the sphere will no longer remain uniform, but that, if the velocity be changing very rapidly, the negative electricity will be heaped around the extremity of the radius of the sphere which is parallel to its acceleration, while, if the velocity of the sphere be small, it will be heaped around the extremity of the radius parallel to the direction of motion. These effects are, of course, only very small for all practicable velocities and accelerations. The original charge on the sphere is supposed to be positive.

In the problems we are about to consider, a spherical conductor charged with electricity is placed in the midst of an insulating medium which stretches to infinity in all directions. In the dielectric there will be what Maxwell calls electric displacements, and, if these vary for any reason, their variations produce the same effects as ordinary electric currents; thus they will give rise to magnetic forces, possess self-induction, and induce currents in the substance of the conductor. The investigation of these effects is the object of the following paper.

The following is the notation we shall employ:—f, g, h are the com-

ponents of electric displacement parallel to the axes of x, y, z respectively; u, v, w are the components of the electric current in the conductor.

F, G, H, are the components of the vector potential of magnetic induction.

K is the specific inductive capacity of the dielectric, σ is the specific resistance of the conductor, μ and μ' are the coefficients of magnetic permeability of the dielectric and the conductor.

ϕ is the potential due to the electricity in the field.

The equations satisfied by these quantities in the dielectric are

$$\frac{4\pi}{K}f = -\frac{dF}{dt} - \frac{d\phi}{dx} \quad \ldots\ldots(1),$$

$$\nabla^2 F = -4\pi\mu \frac{df}{dt} \quad \ldots\ldots(2),$$

with symmetrical equations for g, h, G, H; also

$$\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \quad \ldots\ldots(3),$$

since there is no free electricity in the dielectric.

We have also

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0 \quad \ldots\ldots(4).$$

In the conductor the equations are

$$\sigma u = -\frac{dF}{dt} - \frac{d\phi}{dx} \quad \ldots\ldots(5),$$

with symmetrical equations for v and w.

$$\nabla^2 F = -4\pi\mu' u \quad \ldots\ldots(6),$$

with symmetrical equations for G and H

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0,$$

and

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = -\frac{d\epsilon}{dt} \quad \ldots\ldots(7),$$

where ϵ is the volume density of the free electricity in the conductor.

Thus F may be looked upon as the potential due to a distribution of matter of density $\mu' u$ throughout the conductor and $\mu' \frac{df}{dt}$ throughout the dielectric. Thus F, G, H, and all their differential coefficients will be finite and continuous as long as u, v, w, $\frac{df}{dt}$, $\frac{dg}{dt}$, $\frac{dh}{dt}$ are finite.

We notice also that, if there is no electricity except on the surface of the conductor,

$$\nabla^2 \phi = 0.$$

Differentiating equation (1) with respect to t, we get

$$\frac{4\pi}{K}\frac{df}{dt} = -\frac{d^2F}{dt^2} - \frac{d^2\phi}{dx\,dt} \quad \text{.....(8),}$$

or, substituting for $\frac{df}{dt}$ its value from equation (2), we have

$$-\frac{1}{\mu K}\nabla^2 F = -\frac{d^2F}{dt^2} - \frac{d^2\phi}{dt\,dx} \quad \text{.....(9).}$$

We shall suppose that all the functions concerned vary as ϵ^{ipt} where p may be composite. With this supposition, equation (9) becomes

$$-\frac{1}{\mu K}\nabla^2 F = p^2 F - ip\frac{d\phi}{dx} \quad \text{.....(10).}$$

Since $\nabla^2\phi = 0$, a particular solution of this differential equation is

$$F = \frac{i}{p}\frac{d\phi}{dx}.$$

The complementary function is the solution of the differential equation

$$(\nabla^2 + p^2\mu K)\,F = 0,$$

or, writing $\lambda^2 = p^2\mu K$,

$$(\nabla^2 + \lambda^2)\,F = 0 \quad \text{.....(11).}$$

Let F vary as y_n the surface harmonic of the n^{th} order. Then, if $F = Y_n F'$, where F' is a function of r the distance of a point from the centre of the spherical conductor, we can easily prove that F' must satisfy the differential equation

$$\frac{d^2F'}{dr^2} + \frac{2}{r}\frac{dF'}{dr} + \left(\lambda^2 - \frac{n\,.\,n+1}{r^2}\right)F = 0 \quad \text{.....(12).}$$

This equation has been discussed by various writers. The following are references to some of the more important papers on the subject :—

Stokes, "On the Communication of Vibrations from a Vibrating Body to a surrounding Gas." *Phil. Trans.*, 1868.

Lord Rayleigh, "Theory of Sound." Chap. xvii., vol. II.

Professor C. Niven, "On the Conduction of Heat in Ellipsoids of Revolution." *Phil. Trans.* Part 1. 1880.

Professor C. Niven, "On the Induction of Electric Currents in Infinite Plates and Spherical Shells." *Phil. Trans.* Part II. 1881.

Professor H. Lamb, "On the Vibrations of an Elastic Sphere," and "On the Oscillations of a Viscous Spheroid." *Proc. Lond. Math. Soc.*, 1882.

Professor H. Lamb, "On Electrical Motions in a Spherical Conductor." *Phil. Trans.* Part II. 1883.

V. Helmholtz, Collected Papers, s. 320, Vol. I.

Heine, Kugelfunctionen, s. 240, Vol. I.

The differential equation (12) is satisfied by any of the following functions

$$(\lambda r)^n \left(\frac{1}{\lambda r}\frac{d}{d.\lambda r}\right)^n \frac{\sin \lambda r}{\lambda r}$$

which we shall denote by $S_n(\lambda r)$,

$$\left.\begin{aligned} &(\lambda r)^n \left(\frac{1}{\lambda r}\frac{d}{d.\lambda r}\right)^n \frac{\cos \lambda r}{\lambda r} \\ &(i\lambda r)^n \left(\frac{1}{i\lambda r}\frac{d}{d.i\lambda r}\right)^n \frac{\epsilon^{i\lambda r}}{i\lambda r} \\ &(i\lambda r)^n \left(\frac{1}{i\lambda r}\frac{d}{d.i\lambda r}\right)^n \frac{\epsilon^{-i\lambda r}}{i\lambda r} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The first of these is the only one which remains finite when r vanishes, so it is the solution which must be taken whenever r can vanish.

To determine which solution must be taken outside the sphere when r cannot vanish but can become infinite, we notice that, since the currents fade away, owing to the resistance of the conductor, p will equal $\alpha + i\beta$ where β is positive, therefore λ is of the form $\alpha' + i\beta'$ where β' is positive; now the third of the above solutions is the only one which does not become infinite when r is infinite if λ is of this form. So that, outside the sphere,

$$A\,(i\lambda r)^n \left(\frac{1}{i\lambda r}\frac{d}{d\,(i\lambda r)}\right)^n \frac{\epsilon^{i\lambda r}}{i\lambda r}$$

is the solution for our purpose of the differential equation (12), and this will still be the case when we neglect the resistance of the conductor, as we consider this as a particular case of the more general problem where the resistance is taken into account.

In the conductor we have, by substituting in (5) the value of u from (6),

$$\frac{-\sigma}{4\pi\mu'}\nabla^2 F = -\frac{dF}{dt} - \frac{d\phi}{dx},$$

or, since F varies as ϵ^{ipt},

$$\frac{-\sigma}{4\pi\mu'}\nabla^2 F = -ipF - \frac{d\phi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (14);$$

since $\nabla^2\phi = 0$, a particular solution of this differential equation is

$$F = -\frac{1}{ip}\frac{d\phi}{dx}.$$

The complementary function is the solution of the equation

$$(\nabla^2 + \lambda'^2)\,F = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15),$$

if

$$\lambda'^2 = \frac{-4\pi ip\mu'}{\sigma}.$$

Thus, if $F = Y_n F'$, where Y_n is a surface harmonic, $F' = A' S_n(\lambda' r)$, since F' is finite when $r = 0$.

Let us now consider a case where the mathematical conditions are as simple as possible. Let us suppose that we have a very thin spherical shell made of some material which is a good conductor, and that over this surface the distribution of electricity can be initially expressed by a zonal harmonic of the n^{th} degree Q_n, and that the cause which produced this distribution is suddenly withdrawn, we shall now investigate what will happen to the electricity on the surface of the shell.

We shall suppose that the electrical distribution on the shell can still be expressed by $f(t)\ Q_n$, where $f(t)$ is a function of the time. We shall find that we can satisfy all the conditions of the problem with this assumption.

Let us take the axis of the zonal harmonic Q_n as the axis of z. Let us suppose, as before, that all the quantities involved in this problem vary as ϵ^{ipt}. Then, outside the sphere, ϕ can be written

$$A\epsilon^{ipt}\frac{d^n}{dz^n}\frac{1}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Let $(i\lambda r)^n\left(\frac{d}{i\lambda r\,.\,di\lambda r}\right)\frac{\epsilon^{i\lambda r}}{i\lambda r}$ be denoted by $E_n(\lambda r)$. Since the quantities F, G, H satisfy equations of the form

$$\nabla^2 F+\lambda^2 F=ip\mu K\frac{d\phi}{dx},$$

one solution of these equations is, if B be an arbitrary constant,

$$F=\frac{iA}{p}\epsilon^{ipt}\frac{d^{n+1}}{dz^n dx}\frac{1}{r}+B\epsilon^{ipt}E_{n+1}(\lambda r)\,r^{n+2}\frac{d^{n+1}}{dz^n dx}\frac{1}{r} \quad \ldots\ldots\ldots (17),$$

with symmetrical expressions for G and H; these values however do not satisfy the condition

$$\frac{dF}{dx}+\frac{dG}{dy}+\frac{dH}{dz}=0.$$

If, however, we add to F the term

$$(-1)^{n+1}B\epsilon^{ipt}\frac{n+1\,!}{n}\frac{E_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dx}(r^n Q_n),$$

and make similar additions to the expressions for G and H, we get

$$F=\frac{iA}{p}\epsilon^{ipt}\frac{d^{n+1}}{dz^n dx}\frac{1}{r}+B\epsilon^{ipt}E_{n+1}(\lambda r)\,r^{n+2}\frac{d^{n+1}}{dz^n dx}\frac{1}{r}$$
$$+(-1)^n B\frac{(n+1)\,!}{n}\epsilon^{ipt}\frac{E_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dx}(r^n Q_n) \ldots (18),$$

$$G = \frac{iA}{p}\epsilon^{ipt}\frac{d^{n+1}}{dz^n dy}\frac{1}{r} + B\epsilon^{ipt}E_{n+1}(\lambda r)\, r^{n+2}\frac{d^{n+1}}{dz^n dy}\frac{1}{r}$$
$$+(-1)^n B\frac{(n+1)!}{n}\epsilon^{ipt}\frac{E_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dy}(r^n Q_n) \ldots (19),$$

$$H = \frac{iA}{p}\epsilon^{ipt}\frac{d^{n+1}}{dz^{n+1}}\frac{1}{r} B\epsilon^{ipt}E_{n+1}(\lambda r)\, r^{n+2}\frac{d^{n+1}}{dz^{n+1}}\frac{1}{r}$$
$$+(-1)^n\frac{B(n+1)!}{n}\epsilon^{ipt}\frac{E_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dz}(r^n Q_n) \ldots (20).$$

We can prove that these expressions for F, G, H satisfy the condition

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0,$$

for, since $\nabla^2\frac{1}{r} = 0$, the first terms in the expression for F, G, H satisfy this condition; for the same reason the second terms give

$$B\epsilon^{ipt}\frac{d}{dr}[E_{n+1}(\lambda r)\, r^{n+2}]\frac{d}{dr}\frac{d^n}{dz^n}\frac{1}{r}$$
$$= (-1)^{n+1}B(n+1)!\frac{Q_n}{r^{n+2}}\epsilon^{ipt}\frac{d}{dr}[E_{n+1}(\lambda r)\, r^{n+2}] \ldots (21),$$

and, since $\nabla^2(r^n Q_n) = 0$, the third terms give

$$(-1)^{n+1}B(n+1)!\, r^{n-1} Q_n\frac{d}{dr}\left(\frac{E_{n-1}(\lambda r)}{r^{n-1}}\right) \ldots\ldots\ldots\ldots\ldots (22).$$

Now
$$E_n(\lambda r) = (i\lambda r)^n\left(\frac{1}{i\lambda r}\frac{d}{d(i\lambda r)}\right)^n\frac{\epsilon^{i\lambda r}}{i\lambda r},$$

and therefore

$$= (i\lambda r)^{n-1}\frac{d}{d\,.\,i\lambda r}\frac{E_{n-1}(\lambda r)}{(i\lambda r)^{n-1}} = \frac{r^{n-1}}{i\lambda}\frac{d}{dr}\frac{E_{n-1}(\lambda r)}{r^{n-1}}.$$

Thus
$$E_{n+1}(\lambda r) = (i\lambda r)^n\frac{d}{d\,.\,i\lambda r}\frac{E_n(\lambda r)}{(i\lambda r)^n} \ldots\ldots\ldots\ldots\ldots (23),$$

therefore
$$\frac{d}{d(i\lambda r)}E_{n+1}(\lambda r) + \frac{n+2}{i\lambda r}E_{n+1}(\lambda r)$$
$$= (i\lambda r)^n\left\{\left(\frac{d}{d\,.\,i\lambda r}\right)^2\frac{E_n(\lambda r)}{(i\lambda r)^n} + \frac{2(n+1)}{i\lambda r}\frac{d}{d\,.\,i\lambda r}\frac{E_n(\lambda r)}{(i\lambda r)^n}\right\}$$
$$= E_n(\lambda r) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

since equation (12) can be written

$$\left(\frac{d}{d(i\lambda r)}\right)^2\frac{E_n(\lambda r)}{(i\lambda r)^n} + 2(n+1)\frac{d}{d(i\lambda r)}\frac{E_n(\lambda r)}{(i\lambda r)^n} - \frac{E_n(\lambda r)}{(i\lambda r)^n} = 0.$$

Equation (24) can be written

$$\frac{d}{d(i\lambda r)}\{(i\lambda r)^{n+2}E_{n+1}(\lambda r)\} = -(i\lambda r)^{n+2}E_n(\lambda r),$$

or $$\frac{d}{dr}\{r^{n+2}E_{n+1}(\lambda r)\} = -i\lambda \cdot r^{n+2}E_n(\lambda r) \quad \ldots\ldots\ldots\ldots(25);$$

hence, by equation (23),

$$\frac{d}{dr}\{r^{n+2}E_{n+1}(\lambda r)\} = -r^{2n+1}\frac{d}{dr}\frac{E_{n-1}(\lambda r)}{r^{n-1}} \quad \ldots\ldots\ldots\ldots(26);$$

hence, from (21) and (22), we see that

$$\frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0.$$

In connection with these equations we may remark that

$$\frac{d}{dx}(r^nQ_n) = -xr^{n-2}\frac{dQ_{n-1}}{d\mu} \quad \ldots\ldots\ldots\ldots(27),$$

$$\frac{d}{dy}(r^nQ_n) = -yr^{n-2}\frac{dQ_{n-1}}{d\mu} \quad \ldots\ldots\ldots\ldots(28),$$

$$\frac{d}{dz}(r^nQ_n) = nr^{n-1}Q_{n-1} \quad \ldots\ldots\ldots\ldots(29).$$

Inside the spherical shell,

$$\phi = (-1)^n n!\, A\epsilon^{ipt}\frac{r^n}{a^{2n+1}}Q_n \quad \ldots\ldots\ldots\ldots(30),$$

where a is the radius of the shell.

In the dielectric inside the shell, the differential equations are the same as outside. Since, however, F, G, H remain finite when $r = 0$, we must use the functions $S_n(\lambda r)$ instead of $E_n(\lambda r)$. Thus, inside the shell, we shall have

$$F = (-1)^n\frac{iA}{p}\frac{n!}{a^{2n+1}}\epsilon^{ipt}\frac{d}{dx}(r^nQ_n) + C\epsilon^{ipt}S_{n+1}(\lambda r)\,r^{n+2}\frac{d^{n+1}}{dz^n dx}\frac{1}{r}$$
$$+(-1)^{n+1}C\frac{(n+1)!}{n}\epsilon^{ipt}\frac{S_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dx}(r^nQ_n) \quad \ldots\ldots\ldots(31),$$

$$G = (-1)^n\frac{iA}{p}\frac{n!}{a^{2n+1}}\epsilon^{ipt}\frac{d}{dy}(r^nQ_n) + C\epsilon^{ipt}S_{n+1}(\lambda r)\,r^{n+2}\frac{d^{n+1}}{dz^n dy}\frac{1}{r}$$
$$+(-1)^{n+1}C\frac{(n+1)!}{n}\epsilon^{ipt}\frac{S_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dy}(r^nQ_n) \quad \ldots\ldots\ldots(32),$$

$$H = (-1)^n\frac{iA}{p}\frac{n!}{a^{2n+1}}\epsilon^{ipt}\frac{d}{dz}(r^nQ_n) + C\epsilon^{ipt}S_{n+1}(\lambda r)\,r^{n+2}\frac{d^{n+1}}{dz^{n+1}}\frac{1}{r}$$
$$+(-1)^{n+1}C\frac{(n+1)!}{n}\epsilon^{ipt}\frac{S_{n-1}(\lambda r)}{r^{n-1}}\frac{d}{dz}(r^nQ_n) \quad \ldots\ldots\ldots(33).$$

Since the electrical distribution on the shell keeps changing, there must be surface currents to carry the electricity from one part of the sphere to another, let q be the intensity of the current at any time,

then $2\pi a \sin\theta\, q$ is the quantity of electricity which in unit time crosses the small circle of colatitude θ, so that, if σ be the surface density of the electricity at any point of the shell, we have

$$a\frac{d}{d\theta}(q\sin\theta)\,d\phi\,d\theta + a^2\sin\theta\,d\theta\,d\phi\,\frac{d\sigma}{dt} = 0,$$

therefore

$$\frac{d}{d\theta}(q\sin\theta) + a\sin\theta\,\frac{d\sigma}{dt} = 0.$$

Now

$$\sigma = (-1)^n\frac{2n+1}{4\pi}\,\frac{n!\,K}{a^{n+2}}\,Ae^{ipt}\,Q_n \quad \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

therefore

$$\frac{d}{d\theta}(q\sin\theta) + a\sin\theta\,(-1)^n\frac{2n+1}{4\pi}\,\frac{n!}{a^{n+2}}\,ip\,.\,KAe^{ipt}\,Q_n = 0,$$

therefore

$$q\sin\theta = \frac{(-1)^n\,2n+1\,.\,n!}{4\pi\,.\,a^{n+1}}\,ip\,.\,KAe^{ipt}\int_1^{\mu} Q_n\,d\mu,$$

therefore

$$q = \frac{(-1)^{n+1}(2n+1)(n-1)!}{4\pi\,.\,n+1\,.\,a^{n+1}}\,ip\,.\,KAe^{ipt}\sin\theta\,\frac{dQ_n}{d\mu} \ldots\ldots(35),$$

where $\mu = \cos\theta$.

To find the conditions which must be satisfied by F, G, H, we must remember that F, G, H are the potentials due to distribution of matter whose volume density is finite everywhere, except at the surface of the shell where the surface densities are $\mu q\cos\theta\cos\phi$, $\mu q\cos\theta\sin\phi$, $-\mu q\sin\theta$ respectively. Thus F, G, H are continuous at the surface, and

$$\left(\frac{dF}{dr}\right)\text{ inside} - \left(\frac{dF}{dr}\right)\text{ outside} = 4\pi\mu q\cos\theta\cos\phi \quad \ldots\ldots(36),$$

with symmetrical equations for G and H.

The continuity of F, G, H requires that

$$\frac{iA}{p} + Ba^{n+2}E_{n+1}(\lambda a) = Ca^{n+2}S_{n+1}(\lambda a) \ldots\ldots\ldots\ldots\ldots\ldots (37),$$

and

$$-(n+1)\,BE_{n-1}(\lambda a) = -\frac{iA}{p}\,\frac{n}{a^{n+2}} + (n+1)\,CS_{n-1}(\lambda a) \quad \ldots\ldots(38).$$

Equation (36) gives, since

$$(2n+1)\sin\theta\cos\theta\cos\phi\,\frac{dQ_n}{d\mu} = (-1)^{n+1}\frac{1}{n-1\,!}\,a^{n+2}\,\frac{d^{n+1}}{dz^n dx}\,\frac{1}{r} - \frac{n+1}{a^{n-1}}\,\frac{d}{dx}(r^n Q_n) \ldots(39),$$

$$n\,.\,n-1\,.\,\frac{iA}{p} - (n+1)\,Ca^{n+3}\,\frac{d}{da}\,S_{n-1}(\lambda a) - (n+1)\,Ba^{n+3}\,\frac{d}{da}\,E_{(n-1)}(\lambda a) = \mu ipK\,a^2 A \ldots\ldots\ldots\ldots\ldots\ldots(40),$$

and $$(n+2)\frac{iA}{p}+Ca^{n+3}\frac{d}{da}S_{n+1}(\lambda a)+Ba^{n+3}\frac{d}{da}E_{n+1}(\lambda a)$$

$$=\frac{1}{n+1}ip\mu Ka^2A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41).$$

Equations (37), (38), (40), and (41) are not independent, but any two of them may be deduced from the other two by the help of the relations (23) and (25). The electromotive force along a tangent to the spherical shell in a meridian plane

$$=\cos\theta\cos\phi\left(-\frac{dF}{dt}-\frac{d\phi}{dx}\right)+\cos\theta\sin\phi\left(-\frac{dG}{dt}-\frac{d\phi}{dy}\right)$$

$$-\sin\theta\left(-\frac{dH}{dt}-\frac{d\phi}{dz}\right) \quad \ldots\ldots\ldots\ldots\ldots(42),$$

and this equals σq if σ be the resistance of the shell. If σ be very small compared with pa, σq will be very small, and may be taken equal to zero, and the electromotive force along the tangent in this case must vanish. Substituting for F, G, and H their values from equations (31), we see that the electromotive force is

$$(-1)^{n+1}n!\,ipC\left(S_{n+1}(\lambda a)-\frac{n+1}{n}S_{n-1}(\lambda a)\right)\frac{dQ_n}{d\theta}\ldots\ldots\ldots(43),$$

and for this to vanish, supposing C not to be zero,

$$S_{n+1}(\lambda a)-\frac{n+1}{n}S_{n-1}(\lambda a)=0 \quad \ldots\ldots\ldots\ldots\ldots(44).$$

This equation determines λ and therefore p, the frequency of the vibration executed by the distribution of electricity. Since the values of λa or $\sqrt{\mu K}\,pa$ given by this equation are finite, and since $\sqrt{\mu K}$ is approximately equal to the time taken by light to travel the unit of length, we see that the time of vibration is comparable with the time taken by light to pass over a diameter of the spherical shell. This result justifies the neglect of σ/pa in comparison with the expression (42). The electrical vibrations which take place when the opposite plates of a condenser are connected by a coil whose coefficient of self-induction is considerable, are very much slower than the vibrations we have just been considering, and it would be theoretically possible to arrange a series of electrical systems whose times of vibration would range from those of light to those of sound. In order to get a numerical result, let us suppose $n=1$, so that the density of the electricity on the shell varies as $\cos\theta$; equation (44) gives

$$S_2(\lambda a)=2\,S_0(\lambda a)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

or $$-\frac{\sin(\lambda a)}{\lambda a}-\frac{3\cos\lambda a}{(\lambda a)^2}+\frac{3\sin\lambda a}{(\lambda a)^3}=\frac{2\sin\lambda a}{\lambda a},$$

or $$\tan\lambda a=\frac{\lambda a}{1-\lambda^2 a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(46).$$

I find, by the method of trial and error, that the least root of this equation other than zero [which does not satisfy (45)] is 2·745. There are of course an infinite number of other roots greater than this; the relative importance of the vibrations corresponding to the various roots will depend upon the initial disposition of the field.

The time of vibration corresponding to the root 2·745 is the same as the time of vibration of a ray of light whose wave-length is 1·14 times the diameter of the shell. Since the time of vibration is so small, any disturbance which can give rise to the vibrations must be very sudden.

Still considering the case $n=1$, we find, from equations (37) and (38), that when C is not zero B must vanish; thus, in this case, there are neither magnetic nor electromotive forces outside the shell. This corresponds to the case when the electrified system which initially produced the irregular distribution of electricity over the shell was inside the shell; when this system is suddenly discharged, the electricity which was on the shell vibrates, but no magnetic or electromotive forces are produced outside the shell; the efforts of the displacement currents in the dielectric inside the shell to produce such forces are just counterbalanced by the action of the current in the spherical shell. Since there is no magnetic force outside the shell, the system does not lose any of its energy by propagation into the surrounding dielectric, so that, if it were not for the resistance of the shell, the vibrations would last for ever.

Since $B=0$, we find, from equation (37), that

$$C=\frac{iA}{pS_2(\lambda a)\,a^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(47),$$

or, as $S_2(\lambda a)=2\,S_0(\lambda a)$,

$$=\frac{iA}{2pa^3 S_0(\lambda a)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(48),$$

when $\lambda a=2{\cdot}745$ and $S_0(\lambda a)=\dfrac{\sin\lambda a}{\lambda a}$.

Substituting this value of λa, we find

$$C=-\frac{7.1\ .iA}{2pa^3}=\frac{2\pi i\times 7.1\,\sigma}{3Kp} \quad \ldots\ldots\ldots\ldots\ldots\ldots(50),$$

if σ be the maximum surface density of the electricity on the spherical shell.

Now, inside the shell,

$$\left.\begin{aligned} F &= C\,\epsilon^{ipt} S_2(\lambda r)\, r^3 \frac{d^2}{dx\,dz}\frac{1}{r} \\ G &= C\,\epsilon^{ipt} S_2(\lambda r)\, r^3 \frac{d^2}{dy\,dz}\frac{1}{r} \\ H &= C\,\epsilon^{ipt} S_2(\lambda r)\, r^3 \frac{d^2}{dz^2}\frac{1}{r} + 2C\,\epsilon^{ipt} S_0(\lambda r) - \frac{iA}{pa^3}\epsilon^{ipt} \end{aligned}\right\} \quad \ldots\ldots(51).$$

Therefore, if α, β, γ be the components of the magnetic force, substituting these values for H and G in the equation

$$\mu\alpha = \frac{dH}{dy} - \frac{dG}{dz},$$

we get

$$\mu\alpha = 3C\,\epsilon^{ipt}\frac{y}{r}\frac{d}{dr}S_0(\lambda r).$$

Substituting for C and taking the real part, we get

$$\begin{aligned} \alpha &= -\frac{14.2\pi\sigma}{p\mu K}\sin pt\,\frac{y}{r}\frac{d\,S_0(\lambda r)}{dr} \\ &= -\frac{14.2\pi\sigma}{\sqrt{\mu K}}\sin pt\,\frac{y}{r}\frac{d\,S_0(\lambda r)}{d.\,\lambda r} \\ &= -\frac{14.2\pi\sigma}{\sqrt{\mu K}}\sin pt\,\frac{y}{r}\,S_1(\lambda r) \quad \ldots\ldots(52). \end{aligned}$$

Similarly

$$\beta = \frac{14.2\pi\sigma}{\sqrt{\mu K}}\sin pt\,\frac{x}{r}\,S_1(\lambda r) \quad \ldots\ldots(53),$$

$$\gamma = 0.$$

So that the resultant magnetic force is perpendicular to the meridian plane and

$$= \frac{14.2.\pi\sigma}{\sqrt{\mu K}}\sin pt\,S_1(\lambda r) \quad \ldots\ldots(54).$$

This corresponds to a stationary vibration, and the magnetic force vanishes when $S_1(\lambda r) = 0$. This force could easily be made large enough to measure if its value were stationary; it is the very rapid changes in its value which would make the observations difficult. It seems, however, to afford a promising way of testing Maxwell's theory.

Equation (43) can also be satisfied by making C vanish. This corresponds to the case when the electrified system which produced the irregular distribution of electricity was outside the shell. Since $C = 0$, there are neither electromotive nor magnetic forces inside the shell, so that the currents in the shell produce inside it equal and opposite effects to the displacement currents outside.

In this case, energy is propagated from the shell through the medium

so that the electrical distribution on the shell gradually dies away. Hence we see that the value of λ will be no longer wholly real.

We see, from equations (36) and (37), that the values of λ are given by the equation

$$E_2(\lambda a) + 2E_0(\lambda a) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55),$$

or

$$\epsilon^{i\lambda a}\left\{\frac{i}{(\lambda a)^3} - \frac{i}{\lambda a} - \frac{1}{\lambda^2 a^2}\right\} = 0;$$

therefore

$$\lambda a = \frac{i}{2} \pm \sqrt{\frac{3}{4}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (56),$$

therefore

$$p = \frac{iv}{2a} \pm \sqrt{\frac{3}{4}}\,\frac{v}{a}, \text{ when } v = 1/\sqrt{\mu K},$$

therefore

$$\epsilon^{ipt} = \epsilon^{-\frac{v}{2a}t}\,\epsilon^{\pm i\sqrt{\frac{3}{4}}\frac{v}{a}t}.$$

Since the real negative power to which ϵ is raised is so large, the energy will all be dissipated into space in the course of a few vibrations, so that we could not expect any measurable result in this case.

We shall now proceed to investigate the case of a metal sphere charged with electricity and vibrating in a dielectric, and we shall suppose that the motion is confined to the conductor, and that the sphere moves through the dielectric without putting it in motion. This supposition is, of course, not quite accurate, but it does not lead to any error of importance, and it simplifies the equations very much. Let the centre of the sphere vibrate along the axis of z, and let the period of vibration be $2\pi/p$, so that the velocity of the centre of the sphere is $\omega\epsilon^{ipt}$. Then, if we take rectangular axes passing through the centre of the sphere and moving with it, using the same notation as before, the following equations are true inside the sphere:

$$\sigma u = -b\omega\epsilon^{ipt} - \frac{dF}{dt} - \frac{d\psi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (57),$$

$$\sigma v = a\omega\epsilon^{ipt} - \frac{dG}{dt} - \frac{d\psi}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (58),$$

$$\sigma w = -\frac{dH}{dt} - \frac{d\psi}{dz} \quad \ldots\ldots\ldots\ldots\ldots\ldots (59);$$

therefore

$$-\frac{\sigma}{4\pi\mu'}\nabla^2 F = -b\omega\epsilon^{ipt} - \frac{dF}{dt} - \frac{d\psi}{dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (60),$$

$$-\frac{\sigma}{4\pi\mu'}\nabla^2 G = a\omega\epsilon^{ipt} - \frac{dG}{dt} - \frac{d\psi}{dy} \quad \ldots\ldots\ldots\ldots\ldots\ldots (61),$$

$$-\frac{\sigma}{4\pi\mu'}\nabla^2 H = -\frac{dH}{dt} - \frac{d\psi}{dz} \quad \ldots\ldots\ldots\ldots\ldots\ldots (62),$$

where a, b, c are the components of magnetic induction.

In the dielectric outside the sphere, the equations are, if $\frac{\delta}{\delta t}$ denote partial differentiation with respect to t,

$$\frac{4\pi}{K}f = -\frac{dF}{dt} - \frac{d\psi}{dx} = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ipt}\frac{d}{dz}\right)F - \frac{d\psi}{dz} \quad \ldots\ldots(63),$$

$$\frac{4\pi}{K}g = -\frac{dG}{dt} - \frac{d\psi}{dy} = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ipt}\frac{d}{dz}\right)G - \frac{d\psi}{dy} \quad \ldots\ldots(64),$$

$$\frac{4\pi}{K}h = -\frac{dH}{dt} - \frac{d\psi}{dz} = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ipt}\frac{d}{dz}\right)H - \frac{d\psi}{dz} \quad \ldots\ldots(65),$$

therefore $$-\frac{1}{\mu K}\nabla^2 F = -\left(\frac{\delta}{\delta t} - \omega\epsilon^{ipt}\frac{d}{dz}\right)^2 F - \left(\frac{\delta}{\delta t} - \omega\epsilon^{ipt}\right)\frac{d\psi}{dx} \quad \ldots\ldots(66),$$

with similar equations for G and H.

From the form of these equations we see that, if there is no magnetic field except that produced by the moving sphere, the solution will take the form

$$\psi = \psi_0 + \psi_1\epsilon^{ipt} + \psi_1'\epsilon^{-ipt} + \psi_2\epsilon^{2ipt} + \psi_2'\epsilon^{-2ipt} + \&c.,$$

$$F = F_1\epsilon^{ipt} + F_1'\epsilon^{-ipt} + F_2\epsilon^{2ipt} + F_2'\epsilon^{-2ipt} + \ldots$$

If we substitute these values in the equations, we see that we may put ψ_1', ψ_2', ψ_3', ... F_1', F_2', &c. all equal to zero.

If e be the quantity of electricity on the sphere,

$$\psi_0 = \frac{e}{Kr} \quad \ldots\ldots(67).$$

Equating the coefficient of ϵ^{ipt} to zero in equations (60) and (66), we get $$\frac{-\sigma}{4\pi\mu'}\nabla^2 F_1 = -\frac{dF_1}{dt} - \frac{d\psi_1}{dx}, \text{ inside the sphere} \quad \ldots\ldots(68),$$

with similar equations for G_1 and H_1, and outside the sphere we have

$$-\frac{1}{\mu K}\nabla^2 F_1 = -\frac{\delta^2 F}{\delta t^2} - \frac{d^2\psi_1}{dx\,dt} + \frac{\omega e}{K}\frac{d^2}{dx\,dz}\frac{1}{r} \quad \ldots\ldots(69),$$

with similar equations for G_1 and H_1.

The form of equation (69) suggests that we should put

$$\psi_1 = B\frac{d}{dz}\frac{1}{r} \quad \ldots\ldots(70).$$

The particular integral of equation (69) is

$$F_1 = \frac{iB}{p}\frac{d^2}{dx\,dz}\frac{1}{r} - \frac{\omega e}{Kp^2}\frac{d^2}{dx\,dz}\frac{1}{r}.$$

The complementary function is that solution of the differential equation $$\nabla^2 F_1 + \mu K p^2 F_1 = 0 \quad \ldots\ldots(71)$$

which, when considered as a function of the angular coordinates of

the position of a point, varies as $r^3 \frac{d^2}{dx\,dz}\frac{1}{r}$; it must also correspond to a wave propagated outwards from the sphere; hence we see that, since the time enters through the factor ϵ^{ipt}, the solution is of the form

$$C(i\lambda r)^2 r^3 \frac{d^2}{dx\,dz}\frac{1}{r}\left\{\frac{1}{i\lambda r}\frac{d}{d(i\lambda r)}\right\}^2 \frac{\epsilon^{-i\lambda r}}{i\lambda r} \quad \text{.....(72),}$$

where $\lambda^2 = \mu K p^2$.

We shall denote this solution by

$$C\mathfrak{E}_2(i\lambda r)\, r^3 \frac{d^2}{dx\,dz}\frac{1}{r}.$$

Thus

$$F_1 = C\epsilon^{ipt}\mathfrak{E}_2(i\lambda r)\, r^3 \frac{d^2}{dx\,dz}\frac{1}{r} + \frac{iB\epsilon^{ipt}}{p}\frac{d^2}{dx\,dz}\frac{1}{r} - \frac{\omega e\epsilon^{ipt}}{Kp^2}\frac{d^2}{dx\,dz}\frac{1}{r} \quad \text{...(73),}$$

$$G_1 = C\epsilon^{ipt}\mathfrak{E}_2(i\lambda r)\, r^3 \frac{d^2}{dy\,dz}\frac{1}{r} + \frac{iB\epsilon^{ipt}}{p}\frac{d^2}{dy\,dz}\frac{1}{r} - \frac{\omega e\epsilon^{ipt}}{Kp^2}\frac{d^2}{dy\,dz}\frac{1}{r} \quad \text{...(74),}$$

$$H_1 = C\epsilon^{ipt}\mathfrak{E}_2(i\lambda r)\, r^3 \frac{d^2}{dz^2}\frac{1}{r} - 2C\mathfrak{E}_0(i\lambda r) + \frac{iB\epsilon^{ipt}}{p}\frac{d^2}{dz^2}\frac{1}{r} - \frac{\omega e\epsilon^{ipt}}{Kp^2}\frac{d^2}{dz^2}\frac{1}{r} \quad \text{...(75),}$$

where $\mathfrak{E}_0(i\lambda r) = \frac{\epsilon^{-i\lambda r}}{i\lambda r}$, and is introduced to make the expressions for F_1, G_1, H_1 satisfy the condition

$$\frac{dF_1}{dx} + \frac{dG_1}{dy} + \frac{dH_1}{dz} = 0.$$

Inside the sphere, the differential equations for F_1 and G_1 are of the form

$$-\frac{\sigma}{4\pi\mu}\nabla^2 F_1 = -ipF_1 \quad \text{.....(76),}$$

if $\lambda_1^2 = -4\pi ip\,\mu/\sigma$. The solution of this equation is

$$F_1 = D\epsilon^{ipt} S_2(\lambda_1 r)\, r^3 \frac{d^2}{dx\,dz}\frac{1}{r} \quad \text{.....(77),}$$

where $$S_2(\lambda_1 r) = (\lambda_1 r)^2\left(\frac{1}{\lambda_1 r}\frac{d}{d(\lambda_1 r)}\right)^2 \frac{\sin\lambda_1 r}{\lambda_1 r},$$

as this function satisfies the differential equation, and does not become infinite when $r = 0$. Similarly,

$$G_1 = D\epsilon^{ipt} S_2(\lambda_1 r)\, r^3 \frac{d^2}{dy\,dz}\frac{1}{r} \quad \text{.....(78),}$$

the differential equation for H_1 is

$$-\frac{\sigma}{4\mu}\nabla^2 H_1 = -ipH_1 + \frac{B}{a^3} \quad \text{.....(79),}$$

therefore $H_1 = De^{ipt} S_2(\lambda_1 r)\, r^3 \frac{d^2}{dz^2}\frac{1}{r} + 2De^{ipt} S_0(\lambda_1 r) + \frac{B}{ipa^3} e^{ipt}$ ...(80),

where $S_0(\lambda_1 r) = \frac{\sin \lambda_1 r}{\lambda_1 r}$, and is introduced for the same reason as $\mathfrak{E}_0(i\lambda r)$ in the value of H_1 outside the sphere.

Since the quantities F_1, G_1, H_1 are continuous, we may equate the expressions for their values outside and inside when $r = a$; therefore

$$C\mathfrak{E}_2(i\lambda a) - DS_2(\lambda_1 a) + \frac{iB}{pa^3} = \frac{\omega e}{p^2 a^3 K} \quad \text{.....(81)},$$

$$-2C\mathfrak{E}_0(i\lambda a) - 2DS_0(\lambda_1 a) + \frac{iB}{pa^3} = 0 \quad \text{.....(82)}.$$

Since there are no surface currents in this case, the differential coefficients of F_1, G_1, H_1 with regard to r are continuous; hence we get

$$C\frac{d\mathfrak{E}_2(i\lambda a)}{da} - D\frac{dS_2(\lambda_1 a)}{da} - \frac{3iB}{pa^4} = -\frac{3\omega}{p^2 a^4}\frac{e}{K} \quad \text{.....(83)},$$

$$-2C\frac{d\mathfrak{E}_0(i\lambda a)}{da} - 2D\frac{dS_0(\lambda_1 a)}{da} = 0 \quad \text{.....(84)}.$$

We can show, as before, that equations (81), (82), (83), and (84) are not independent. Taking the first, second, and fourth of these, we find

$$C\Delta = -\frac{\omega e}{p^2 a^3 K}\frac{dS_0(\lambda a)}{da} \quad \text{.....(85)},$$

$$+D\Delta = \frac{\omega}{p^2 a^3}\frac{e}{K}\frac{d\mathfrak{E}_0(i\lambda a)}{da} \quad \text{.....(86)},$$

$$\frac{iB}{pa^3}\Delta = -2\frac{\omega}{p^2 a^3}\frac{e}{K}\left\{\mathfrak{E}_0(i\lambda a)\frac{dS_0(\lambda_1 a)}{da} - S_0(\lambda_1 a)\frac{d\mathfrak{E}_0(i\lambda a)}{da}\right\} \text{...(87)},$$

where Δ denotes the determinant

$$\begin{vmatrix} \mathfrak{E}_2(i\lambda a), & S_2(\lambda_1 a), & 1 \\ -2\mathfrak{E}_0(i\lambda a), & 2S_0(\lambda_1 a), & +1 \\ -\frac{d\mathfrak{E}_0(i\lambda a)}{da}, & \frac{dS_0(\lambda_1 a)}{da}, & 0 \end{vmatrix}.$$

Since $\lambda^2 = \mu K p^2$ and $\mu K = 1/v^2$, where v is approximately equal to the velocity of light, we see that, unless the sphere be vibrating so rapidly that its period is comparable with the time light takes to travel over a diameter of the sphere, λa will be very small.

Now,
$$\mathfrak{E}_2(i\lambda a) = \frac{3\epsilon^{-i\lambda a}}{(i\lambda a)^3} + \frac{3\epsilon^{-i\lambda a}}{(i\lambda a)^2} + \frac{\epsilon^{-i\lambda a}}{i\lambda a},$$

$$\mathfrak{E}_0(i\lambda a) = \frac{\epsilon^{-i\lambda a}}{i\lambda a},$$

and, as $i\lambda a$ is very small, the first term of $\mathfrak{E}_2(i\lambda a)$ is very large com-

pared with the others, and also with $\mathfrak{E}_0(i\lambda a)$ and $\frac{d\mathfrak{E}_0(i\lambda a)}{da}$. Thus approximately

$$\Delta = -\frac{3\epsilon^{-i\lambda a}}{(i\lambda a)^3}\frac{dS_0(\lambda_1 a)}{da} \quad \ldots\ldots(88).$$

Therefore

$$C = \tfrac{1}{3}(i\lambda a)^3 \epsilon^{i\lambda}{}_a \frac{\omega e}{p^2 a^3 K} \quad \ldots\ldots(89).$$

The term

$$2C\mathfrak{E}_0(i\lambda r)\,\epsilon^{ipt} = \frac{2}{3}\frac{\omega e (i\lambda a)^3}{p^2 a^3 K}\frac{\epsilon^{i[pt-\lambda(r-a)]}}{i\lambda r} \quad \ldots\ldots(90).$$

The term $C\mathfrak{E}_2(i\lambda r)\,\epsilon^{ipt}$

$$= \frac{1}{3}\frac{\omega e (i\lambda a)^3}{p^2 a^3 K}\epsilon^{i[pt-\lambda(r-a)]}\left\{\frac{3}{(i\lambda r)^3}+\frac{3}{(i\lambda r)^2}+\frac{1}{i\lambda r}\right\} \quad \ldots\ldots(91).$$

Thus, outside the sphere,

$$F_1 = \frac{1}{3}\frac{\omega e (i\lambda a)^3}{p^2 a^3 K}\epsilon^{i[pt-\lambda(r-a)]}\left\{\frac{3}{(i\lambda r)^3}+\frac{3}{(i\lambda r)^2}+\frac{1}{(i\lambda r)}\right\} r^3 \frac{d^2}{dx\,dz}\frac{1}{r}$$
$$+\frac{iB}{p}\frac{d^2}{dx\,dz}\frac{1}{r}-\frac{\omega e}{p^2 a^3 K}\frac{d^2}{dx\,dz}\frac{1}{r} \quad \ldots\ldots(92),$$

with a similar expression for G_1;

$$H_1 = \frac{1}{3}\frac{\omega e (i\lambda a)^3}{p^2 a^3 K}\epsilon^{i[pt-\lambda(r-a)]}\left\{\frac{3}{(i\lambda r)^3}+\frac{3}{(i\lambda r)^2}+\frac{1}{i\lambda r}\right\} r^3 \frac{d^2}{dz^2}\frac{1}{r}$$
$$-\frac{2}{3}\frac{\omega e (i\lambda a)^3}{p^2 a^3 K}\frac{\epsilon^{i[pt-\lambda(r-a)]}}{i\lambda r}+\frac{iB}{p}\frac{d^2}{dz^2}\frac{1}{r}-\frac{\omega e}{p^2 a^3 K}\frac{d^2}{dz^2}\frac{1}{r} \quad \ldots\ldots(93).$$

If a_1, b_1, c_1 be those components of magnetic induction outside the sphere which vary as ϵ^{ipt},

$$a_1 = \frac{dH_1}{dy} - \frac{dG_1}{dz}.$$

Differentiating, we find

$$a_1 = \frac{\omega e}{p^2 a^3 K}(i\lambda a)^3 \epsilon^{i[pt-\lambda(r-a)]}\left\{\frac{1}{i\lambda}+r\right\}\frac{y}{r^3} \quad \ldots\ldots(94).$$

Taking the real part, since $\lambda^2 = p^2\mu K$,

$$a_1 = -\mu\omega e\{\cos[pt-\lambda(r-a)]-\lambda r\sin[pt-\lambda(r-a)]\}\frac{y}{r^3}$$
$$= \mu\omega e\frac{d}{dy}\frac{\cos[pt-\lambda(r-a)]}{r} \quad \ldots\ldots(95).$$

Similarly,

$$b_1 = \mu\omega e\{\cos[pt-\lambda(r-a)]-\lambda r\sin[pt-\lambda(r-a)]\}\frac{x}{r^3}$$
$$= -\mu\omega e\frac{d}{dx}\frac{\cos[pt-\lambda(r-a)]}{r} \quad \ldots\ldots(96);$$
$$c_1 = 0.$$

If $p = 0$, or if the velocity be constant, then $\lambda = 0$, and

$$a_1 = -\mu \frac{\omega e y}{r^3},$$

$$b_1 = \mu \frac{\omega e x}{r^3}$$

$$c_1 = 0,$$

or the moving charged sphere produces the same magnetic forces as a unit length of a current of strength ωe.

The expressions

$$\mathfrak{a}_1 = m \frac{d}{dy} \frac{\cos(pt - \lambda r)}{r},$$

$$\mathfrak{b}_1 = -m \frac{d}{dx} \frac{\cos(pt - \lambda r)}{r},$$

$$\mathfrak{c}_1 = 0,$$

satisfy equations of the form

$$\nabla^2 \mathfrak{a}_1 = \mu K \frac{d^2 \mathfrak{a}}{dt^2},$$

and they make the line integral

$$\int (\mathfrak{a}_1 dx + \mathfrak{b}_1 dy + \mathfrak{c}_1 dz),$$

taken round a circuit for which r is indefinitely small,

$$= 2\pi m \mu \cos pt;$$

hence these expressions are the x, y, z components of the magnetic induction due to an element of current of unit length and intensity $m \cos pt$ flowing along the axis of z. Comparing these expressions with equations (95) and (96), we see that, if r be large compared with a, so that we may write λr instead of $\lambda(r - a)$, the magnetic force produced by a sphere whose radius is a, charged with a quantity e of electricity and moving along the axis of z, so that its velocity is expressed by $\omega \cos pt$, is the same as that produced by an element of current along the axis of z of unit length and intensity $\omega e \cos pt$, so that the equivalence of the moving sphere and the current still holds when the velocity of the sphere is variable.

We will now calculate the mechanical force on the sphere, when it is placed in a strong magnetic field where the magnetic induction is parallel to the axis of y and is equal to b.

The mechanical force parallel to the axis of x per unit of volume is

$$-wb + eP \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (97),*$$

* See Prof. G. F. Fitzgerald, "On Electromagnetic Actions produced by the Motion of the Earth," *Trans. Roy. Soc.*, Dublin, 1883.

where P is the electromotive force parallel to the axis of x. Now

$$-wb = \frac{1}{\sigma}\left(\frac{dH}{dt} + \frac{d\psi}{dz}\right) b.$$

The whole force parallel to the axis of x is therefore

$$\frac{b}{\sigma}\iiint\left(\frac{dH}{dt} + \frac{d\psi}{dz}\right) dx\,dy\,dz + \iiint \rho P\,dx\,dy\,dz.$$

Substituting the value of H given in equation (80), and retaining only those terms which do not vanish on integration, we find that the first member of this expression equals

$$2ipD\frac{b}{\sigma}\epsilon^{ipt}\int_0^a S_0(\lambda_1 r)\,4\pi r^2 dr$$

$$= 8i\pi pD\frac{b}{\sigma}\epsilon^{ipt}\frac{(\sin\lambda_1 a - \lambda_1 a\cos\lambda_1 a)}{\lambda_1^3}$$

$$= 8i\pi pD\frac{ba^2}{\sigma\lambda_1^2}\epsilon^{ipt}\frac{dS_0(\lambda_1 a)}{da} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (98);$$

but, from equation (86),

$$\frac{3D}{(i\lambda a)^3}\epsilon^{-i\lambda a}\frac{dS_0(\lambda_1 a)}{da} = \frac{\omega e}{p^2 a^3 K}\frac{\epsilon^{-i\lambda a}}{i\lambda a^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (99),$$

approximately, if λa be small.

Substituting this value for D, (98) becomes

$$-\frac{8i\pi}{3\sigma\lambda_1^2}p\mu b\omega\epsilon^{ipt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (100).$$

Now $\lambda_1^2 = -\frac{4\pi}{\sigma}ip\mu'$, therefore (100) becomes, taking only the real part,

$$\frac{2}{3}\frac{\mu}{\mu'}e\omega\, b\cos pt \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (101).$$

Since

$$P = -b\omega\epsilon^{ipt} - \frac{dF}{dt} - \frac{d\psi}{dx},$$

$$\iiint \rho P\,dx\,dy\,dz = -eb\omega\epsilon^{ipt},$$

or, taking the real part, $= -eb\omega\cos pt$ ..... (102).

Thus the force on the sphere

$$= eb\,\omega\cos pt\left\{\frac{2}{3}\frac{\mu}{\mu'} - 1\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (103).$$

If $\mu = \mu'$, the force $= -\frac{1}{3}eb\,\omega\cos pt$, or, *i.e.*, one-third of the force on an element of unit length of current whose intensity equals $\omega\cos pt$.

We shall now discuss the value of the electrostatic potential. From

equation (87) we get, supposing λa to be small,

$$\frac{3iB}{pa^3}\frac{\epsilon^{-i\lambda a}}{(i\lambda a)^3}\frac{dS_0(\lambda_1 a)}{da} = \frac{2\omega e}{p^2a^3K}\frac{\epsilon^{-i\lambda a}}{i\lambda a}\left\{\frac{dS_0(\lambda_1 a)}{da} + \frac{S_0(\lambda_1 a)}{a}\right\},$$

therefore
$$iB = -\tfrac{2}{3}\omega e\mu p a^2\left\{1+\frac{S_0(\lambda_1 a)}{\lambda_1 a\, S_1(\lambda_1 a)}\right\} \quad \text{.....(104).}$$

If $\lambda_1 a$ be large,
$$iB = -\tfrac{2}{3}\omega e\mu p a^2,$$

therefore the real part of $B\epsilon^{ipt}\frac{d}{dz}\frac{1}{r}$

$$= -\tfrac{2}{3}\mu e\omega a^2 p\sin pt\,\frac{d}{dz}\frac{1}{r} \quad \text{.....(105);}$$

therefore the maximum surface density of the electricity represented by this distribution

$$= \frac{\mu K pe\omega}{2\pi a},$$

$$= \frac{\mu K e}{2\pi a} \text{ (acceleration of the sphere).}$$

From (105) we see that, if an electrified sphere be moving in a dielectric, the distribution of electricity over the sphere will not remain uniform, but the positive electricity will be heaped up towards the extremity of the radius vector, which points in the direction opposite to that of the acceleration. The original charge on the sphere is supposed to be positive.

If $\lambda_1 a$ be small, we get, from equation (104),

$$iB = \frac{2\omega e\mu p}{\lambda_1^2},$$

or, since $\lambda_1^2 = -\frac{4\pi ip\mu_1}{\sigma}$,

$$B = \frac{1}{2\pi}\frac{\sigma\mu}{\mu_1}\omega e.$$

Therefore the real part of $B\epsilon^{ipt}\frac{d}{dz}\frac{1}{r}$ is

$$\frac{\omega e}{2\pi}\frac{\mu}{\mu_1}\sigma\cos pt\,\frac{d}{dz}\frac{1}{r};$$

the maximum surface density of the electricity represented by this distribution is

$$\frac{3\omega eK}{8\pi^2}\frac{\mu}{\mu_1}\frac{\sigma\cos pt}{a^3},$$

$$= \frac{3e\sigma}{8\pi^2}\frac{\mu K}{\mu'}\frac{\text{(velocity of the sphere)}}{a^3},$$

and the electricity is heaped towards the extremity of the radius which points in the direction opposite to the direction of motion of the sphere.

If we substitute the values of F_1, G_1, H_1, which we have just found in equations (63), (64), &c., we shall get equations to find F_2, G_2, H_2, which we may solve in the same way. We shall find, however, that these quantities are small compared with F_1, G_1, H_1.

Thursday, May 8th, 1884.

Prof. HENRICI, F.R.S., President, in the Chair.

Mr. J. Brill, B.A., Scholar of St. John's College, Cambridge, Lecturer in Mathematics at the University College, Aberystwith, was elected a Member; and Prof. Cremona, of Rome, was admitted into the Society.

The following communications were made :—

"Motion of a Network of Particles with some Analogies to Conjugate Functions," Dr. E. J. Routh, F.R.S.

"On a Subsidiary Elliptic Function $pm\,(u, k)$," Mr. J. Griffiths.

"On the Homogeneous Equation of a Plane Section of a Geometrical Surface," J. J. Walker, F.R.S.

"On a Birational Transformation of Space of Three Dimensions, of the Sixth Degree, the inverse of which is of the fifth," Prof. Cremona.

"Some Properties of Two Lines in the Plane of a Triangle," Mr. R. Tucker.

The following presents were received :—

"Educational Times," May, 1884.

"Integral Calculus," by B. Williamson, 5th ed., 8vo; London, 1884.

"Johns Hopkins University Circulars," Vol. 3, No. 29.

"Mittheilungen der Mathematischen Gesellschaft in Hamburg," No. 4; Hamburg, 1884.

"Beiblätter zu den Annalen der Physik und Chemie," B. 8, St. 3.

"Théorie Elémentaire des Séries Récurrentes," by M. Maurice D'Ocagne. (Extracted from "Nouvelles Annales de Mathématiques.")

"Sur un Algorithme Algébrique," par M. Maurice D'Ocagne. 8vo, pamphlet.

"Sur l'Enveloppe de certaines Droites Variables," by Maurice D'Ocagne, 8vo, pamphlet.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. xviii., Nos. 2, 3, 4, and 5, 1883; T. xix., No. 1, 1884; Harlem, 1884.

"Bulletin des Sciences Mathématiques et Astronomiques," t. viii., February and March, 1884.

"Atti della R. Accademia dei Lincei," Vol. 8, F. 7, 8, 9, 10; 1884.

"Bulletin Astronomique et Météorologique de Rio de Janeiro," Sept. 1881, and Nov., 1883.

"Acta Mathematica," Vol. iii., Pt. 4.

"Programm des Gymnasiums zu Recklinghausen," liv. Schuljahr 1883—4. ("Untersuchungen über ähnliche Punktreihen auf den Seiten eines Dreiecks und auf deren Mittelsenkrechten, sowie über Kongruente Strahlenbüschel aus den Ecken desselben;" ein Beitrag zur Geometrie des Brocardschen Kreises vom Oberlehrer A. Artzt.)

On a Subsidiary Elliptic Function pm (u, k).*

By John Griffiths, M.A.

[*Read May 8th*, 1884.]

I think there is an advantage in using a subsidiary elliptic function, pm (u, k), as I propose to write it, in conjunction with the ordinary am (u, k).

The theory of transformation leads in a very remarkable way to addition-theorems as regards both these functions.

For example, in a recent communication to the Society, it was shown by me that

$$\text{am}\,(3u, k) = 3\,\text{am}\,(u, k) - \text{am}\,(Mu, \lambda) + \text{am}\,(M_1 u, \lambda_1) + \text{am}\,(M_2 u, \lambda_2) + \text{am}\,(M_3 u, \lambda_3);$$

where λ, λ_1, λ_2, λ_3 are the roots of the modular equation in the cubic transformation. This being so, it can be proved that there is the correlative addition-theorem

$$\text{pm}\,(3u, k) = 3\,\text{pm}\,(u, k) - \text{pm}\,(M_1 u, \lambda) + \ldots.*$$

1. *Definition of pm* (u, k).

The new function is derived from am (u, k) by the equation

$$\sin \text{pm}\,(u, k) = k \sin \text{am}\,(u, k) = k\,\text{sn}\,(u, k),$$

or, say,

$$\text{sm}\,(u, k) = k\,\text{sn}\,(u, k).$$

Differentiating this with regard to u, we have

$$\cos \text{pm}\, u \frac{d}{drt} \text{pm}\, u = k\,\text{cn}\, u\,\text{dn}\, u,$$

or, since $\cos \text{pm}\, u = \text{dn}\, u$,

$$\frac{d}{drt} \text{pm}\, u = k\,\text{cn}\, u.$$

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* The notation "pm (u, k)" has been adopted at Prof. Cayley's suggestion. Since $\sin \text{pm}\,(u, k) = \sin \text{am}\left(ku, \frac{1}{k}\right)$, he thinks that pm $(u, k) = \text{am}\left(ku, \frac{1}{k}\right)$ is the best definition of pm (u, k).

If $\sin\phi = k\sin\theta$, it follows at once that

$$\frac{d\phi}{\sqrt{1-\frac{1}{k^2}\sin^2\phi}} = k\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}};$$

hence, putting $\theta = \operatorname{am} u$, we have

$$\sin\operatorname{am}\left(ku, \frac{1}{k}\right) = k\sin\operatorname{am}(u, k) = k\operatorname{sn}(u, k) = \operatorname{sm}(u, k),$$

where $$u = \int_0^{\phi}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The definition, then, gives the following relations, viz.,

$$\left.\begin{aligned}
&\operatorname{sn}\left(ku, \frac{1}{k}\right) = \operatorname{sm}(u, k) = k\operatorname{sn}(u, k)\\
&\operatorname{sn}(u, k) = \operatorname{sm}\left(ku, \frac{1}{k}\right)\\
&\operatorname{cn}\left(ku, \frac{1}{k}\right) = \operatorname{cm}(u, k) = \operatorname{dn}(u, k)\\
&\operatorname{cn}(u, k) = \operatorname{cm}\left(ku, \frac{1}{k}\right) = \operatorname{dn}\left(ku, \frac{1}{k}\right)
\end{aligned}\right\}.$$

Again, since

$$\int_0^{\pi-\phi}\frac{d\phi}{\sqrt{k^2-\sin^2\phi}} = \int_0^{\pi}\frac{d\phi}{\sqrt{k^2-\sin^2\phi}} - \int_0^{\phi}\frac{d\phi}{\sqrt{k^2-\sin^2\phi}},$$

and $$\int_0^{\pi}\frac{d\phi}{\sqrt{k^2-\sin^2\phi}} = 2\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{k^2-\sin^2\phi}} = 2\int_0^{\sin^{-1}\left(\frac{1}{k}\right)}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

$$= 2(K+iK'),$$

it follows that

$$\pi - \operatorname{pm}(u, k) = \operatorname{pm}(2K+2iK'-u, k).$$

It will be presently shown that there is an expression for $\operatorname{pm}\frac{2K}{\pi}x$, in terms of q-functions, viz.,

$$\sin^{-1}\left\{\operatorname{sm}\frac{2K}{\pi}x\right\} = 2\Sigma(-)^{m-1}\tan^{-1}\left\{\frac{2q^{m-\frac{1}{2}}\sin x}{1-q^{2m-1}}\right\}, \quad \left(\begin{matrix} q = e^{-\frac{\pi K'}{K}} \\ x = \frac{u}{2K}\end{matrix}\right).$$

One of the minor advantages of the above notation is that

$$\operatorname{dn} u \equiv \cos\operatorname{pm} u$$

is clearly seen to be a kind of cosine function.

2. *Application of the foregoing results to transformation formulæ.* The first transformation is

$$\sin^{-1}\{\mathrm{sn}\,(Mu, \lambda)\} = \sin^{-1}\{\mathrm{sn}\,(u, k)\} + 2\Sigma \tan^{-1}\left\{\mathrm{dn}\left(\frac{4sK}{n}, k\right) \cdot \frac{\mathrm{sn}\,(u, k)}{\mathrm{cn}\,(u, k)}\right\};$$

change λ into $\frac{1}{\lambda}$, k into $\frac{1}{k}$, M into $M\frac{\lambda}{k}$, u into ku, and K into kK,

then

$$\sin^{-1}\left\{\mathrm{sn}\left(M\lambda u, \frac{1}{\lambda}\right)\right\}$$

$$= \sin^{-1}\left\{\mathrm{sn}\left(ku, \frac{1}{k}\right)\right\} + 2\Sigma \tan^{-1}\left\{\mathrm{dn}\left(\frac{4skK}{n}, \frac{1}{k}\right) \cdot \frac{\mathrm{sn}\left(ku, \frac{1}{k}\right)}{\mathrm{cn}\left(ku, \frac{1}{k}\right)}\right\},$$

i.e., from the above results,

$$\sin^{-1}\{\mathrm{sm}\,(Mu, \lambda)\} = \sin^{-1}\{\mathrm{sm}\,(u, k)\} + 2\Sigma \tan^{-1}\left\{\mathrm{cn}\left(\frac{4sK}{n}, k\right) \cdot \frac{k\,\mathrm{sn}\,(u, k)}{\mathrm{dn}\,(u, k)}\right\},$$

or $$\mathrm{pm}\,(Mu, \lambda) = \mathrm{pm}\,(u, k) + 2\Sigma \tan^{-1}\left\{k\,\mathrm{cn}\left(\frac{4sK}{n}, k\right) \cdot \frac{\mathrm{sn}\,(u, k)}{\mathrm{dn}\,(u, k)}\right\}.$$

Differentiating this in regard to u, and taking

$$\frac{d}{du}\,\mathrm{pm}\,(u, k) = k\,\mathrm{cn}\,(u, k),$$

$$\frac{d}{du}\,\frac{\mathrm{sn}\,(u, k)}{\mathrm{dn}\,(u, k)} = \frac{\mathrm{cn}\,(u, k)}{\mathrm{dn}^2\,(u, k)},$$

we have

$$M\frac{\lambda}{k} \cdot \mathrm{cn}\,(Mu, \lambda) = \mathrm{cn}\,(u, k)\left\{1 + 2\Sigma \frac{\mathrm{cn}\,\frac{4sK}{n}}{1 - k^2\,\mathrm{sn}^2\,\frac{4sK}{n}\,\mathrm{sn}^2\,(u, k)}\right\},$$

which is one of the ordinary formulæ.

Similarly, the second transformation is

$$(-)^{\frac{1}{2}(n-1)}\,\mathrm{pm}\,(M_1 u, \lambda_1) = \mathrm{pm}\,(u, k) + 2\Sigma\,(-)^s \tan^{-1}\left\{k\,\mathrm{cn}\left(\frac{2siK'}{n}, k\right) \cdot \frac{\mathrm{sn}\,(u, k)}{\mathrm{dn}\,(u, k)}\right\},$$

where $\mathrm{pm}\,(u, k) = \sin^{-1}\mathrm{sm}\,(u, k) = \sin^{-1}\{k\,\mathrm{sn}\,(u, k)\}$, &c.

Here change λ_1 into k and k into λ, M into N_1, then

$$(-)^{\frac{1}{2}(n-1)}\,\mathrm{pm}\,(N_1 u, k) = \mathrm{pm}\,(u, \lambda) + 2\Sigma\,(-)^s \tan^{-1}\left\{\lambda\,\mathrm{cn}\left(\frac{2si\Lambda'}{n}, \lambda\right) \cdot \frac{\mathrm{sn}\,(u, \lambda)}{\mathrm{dn}\,(u, \lambda)}\right\};$$

change u into Mu, where $M = \frac{n}{N_1}$, hence

$$(-)^{\frac{1}{2}(n-1)} \operatorname{pm}(nu, k)$$

$$= \operatorname{pm}(Mu, \lambda) + 2\Sigma (-)^s \tan^{-1}\left\{\lambda \operatorname{cn}\left(\frac{2si\Lambda'}{n}, \lambda\right) \cdot \frac{\operatorname{sn}(Mu, \lambda)}{\operatorname{dn}(Mu, \lambda)}\right\}.$$

Lastly, change u into $\frac{u}{n}$, and make n infinite, *i.e.*, make

$$\lambda = 0, \quad \frac{M}{n} = \frac{\pi}{2K}, \quad \frac{\Lambda'}{n} = \frac{\pi K'}{2K}, \quad \operatorname{pm}(u, 0) = 0;$$

also $$\lambda \operatorname{cn}\left(\frac{2si\Lambda'}{n}, \lambda\right) = i \operatorname{dn} \frac{(2t-1)\, i\Lambda'}{n} \div \operatorname{sn} \frac{(2t-1)\, i\Lambda'}{n},$$

where t is an integer extending from 1 to $\frac{1}{2}(n-1)$; then we have, ultimately, the formula

$$\operatorname{pm}(u, k) = \sin^{-1}\{k \operatorname{sn}(u, k)\} = 2\Sigma (-)^{m-1} \tan^{-1}\left\{\frac{2q^{m-\frac{1}{2}} \sin \frac{\pi u}{2K}}{1-q^{2m-1}}\right\},$$

if $$q = e^{-\frac{\pi K'}{K}}.$$

This is the correlative of Jacobi's expression in q-functions for $\operatorname{am}(u, k)$.

If we differentiate the expression just obtained with regard to x, then

$$\operatorname{cn} \frac{2Kx}{\pi} = \frac{2\pi}{kK} \cos x \, \Sigma \left\{(-)^{m-1} \frac{q^{m-\frac{1}{2}}(1-q^{2m-1})}{1-2q^{2m-1}\cos 2x + q^{4m-2}}\right\},$$

which is one of Jacobi's formulæ.

In a similar way, if we are given that there is an addition-theorem for a transformation of an odd degree

$$\operatorname{am}(nu, k) = \pm n \operatorname{am}(u, k) + \Sigma \operatorname{am}(Mu, \lambda),$$

the correlative one is obtained by changing the function am into pm.

3. The above examples are, I think, sufficient to show that the two functions are intimately connected together, and that there would be an advantage in using some such symbol as $\operatorname{pm}(u, k)$.

I write down the quadric transformations in the new notation, viz.,

$$\left.\begin{aligned} \operatorname{am}(u, k) + \operatorname{pm}(u, k) &= 2 \operatorname{am}\left(\frac{\Gamma u}{2K}, \gamma\right) \\ \pi - \operatorname{am}(u, k) + \operatorname{pm}(u, k) &= 2 \operatorname{am}\left(\Gamma - \frac{\Gamma u}{2K}, \gamma\right) \end{aligned}\right\},$$

where $$\Gamma = (1+k)K \quad \text{and} \quad \gamma = \frac{2\sqrt{k}}{1+k}.$$

The formula for a transformation from a modulus k to the com-

plementary one, k', is

$$\text{pm}\,(K+iu,\, k)+\text{pm}\,(K'-u,\, k') = \frac{\pi}{2}.$$

As regards transformation in general, the conclusion at which I have arrived is, that the formula for an odd prime number n, viz.,

$$\text{am}\,(nu,\, k) = \pm\, n\,\text{am}\,(u,\, k)+\Sigma\,\text{am}\,(Mu,\, \lambda),$$

together with the above quadric ones, is sufficient for a transformation of any degree whatever.

It may be mentioned that the quadric equations can be presented in different forms; one of these is

$$2\,\text{am}\,(u,\, k) = \text{am}\left(\overline{1+k'}u,\, \frac{1-k'}{1+k'}\right)+\text{pm}\left(\overline{1+k'}u,\, \frac{1-k'}{1+k'}\right).$$

As an illustration of the use of these formulæ, I append the following result.

Let $$2\,\text{am}\,(6u,\, k) = \text{am}\left(3Nu,\, \frac{1-k'}{1+k'}\right)+\text{pm}\left(3Nu,\, \frac{1-k'}{1+k'}\right),$$

where $$N = 2\,(1+k');$$

then, by the triplication formula, the right-hand side of the equation may be broken up into a series of am- and pm-functions of the type-form am (Mu, λ), &c., where the modular relation is that of the cubic transformation, viz.,

$$\sqrt{\lambda k_1}+\sqrt{\lambda' k_1'} = 1, \text{ if } k_1 = \frac{1-k'}{1+k'} \text{ and } k_1' = \frac{2\sqrt{k'}}{1+k'}.$$

The modular equation in question reduces to

$$\{(1+\lambda)(1+k')-2\}^4 = 64\lambda k'\lambda'^2 k^2,$$

and agrees with the result previously given by me from the transformation-equation $y = \sin(A+B+C)$, where

$$\sin A = \frac{(1+k')\,x\sqrt{1-x^2}}{\sqrt{1-k^2x^2}}, \quad \cos B = \frac{1-(1+\beta'^2)\,x^2}{1-\beta^2x^2}, \text{ &c.}$$

Hence, for the analogue of the trigonometrical function $\sin 6u$, the modular equation has been deduced from the general formulæ as above, and in a similar way that corresponding to any number whatever may be arrived at from those of the primes into which the number can be resolved.

On the Homogeneous Equation of a Plane Section of a Geometrical Surface. *By* J. J. Walker.

[*Read May 8th*, 1884.]

1. The formation of the equation to a plane section of a geometrical surface (given by an equation in rectangular Cartesian coordinates) as referred to rectangular axes in that plane, appears to have been first given by Leroy, §§ 85 to 88. Mr. Whitworth, in his "Trilinear Coordinates," 1866, p. 161, formed the trilinear equation; the consideration of the general question arising out of the particular case of the determination of the equation of the section of a right cone.

Two years later, in the *Nouvelles Annales*, June, 1868, M. Housel showed how to form the Cartesian equation to a plane section of a surface, referred to axes with any angles of ordination, the axes of the plane section being two of the three lines in which its plane meets the coordinate planes.

The object of the present paper is to complete the treatment of the question, by forming the trilinear equation to the section, whatever the angles of ordination of the Cartesian axes to which the surface is referred, and then to deal with the case in which the equation to the surface is expressed in a homogeneous form, either by means of tetrahedral or quadriplanar coordinates.

2. Suppose the surface referred to Cartesian axes having any angles of ordination, and $lx+my+nz=p$ to be any plane intersecting the axes OX, OY, OZ in A, B, C; then

$$OA=p/l, \quad OB=p/m, \quad OC=p/n,$$

and if p be the perpendicular from O on the plane, then l, m, n will be the cosines of the angles which p makes with the axes; and

$$\begin{aligned} BC^2 &= OB^2+OC^2-2OB\,.\,OC\cos YOZ \\ &= p^2m^{-2}+p^2n^{-2}-2p^2m^{-1}n^{-1}\cos YOZ, \end{aligned}$$

$$mnBC/p=(m^2+n^2-2mn\cos YOZ)^{\frac{1}{2}}.$$

Now, if x', x_0 are the perpendiculars from any point in the plane ABC, say P, and from A respectively on BC,

$$x:x'=OA \text{ or } p/l:x_0,$$

or $$x : x' \,.\, BC = p^2 : lpx_0 \,.\, BC$$
$$= mn : p \sin O,$$

since $px_0 BC = 6$ vol. $OABC = OA \,.\, OB \,.\, OC \sin O = p^3 \sin O / lmn$;

hence $x = mnx' \,.\, BC / p \sin O = (m^2+n^2-2mn \cos YOZ)^{\frac{1}{2}} / \sin O.$

Similarly $$y = y' (n^2+l^2-2nl \sin ZOX)^{\frac{1}{2}} / \sin O,$$
$$z = z' (l^2+m^2-2lm \sin XOY)^{\frac{1}{2}} / \sin O.$$

3. If $F(xyz) = 0$ be the equation of any geometrical surface referred to the axes OX, OY, OZ, the substitution of the above values for x, y, z will plainly give the trilinear equation of the section made by the plane $lx+\ldots=p$, the result being made homogeneous by means of the factor $$xx_0^{-1}+yy_0^{-1}+zz_0^{-1} = 1.$$

It remains to express x_0, y_0, z_0 in terms of l, m, n, p, and the angles of ordination.

It has just been shown that

$$x_0 = p^2 \sin O / lmn\, BC$$
$$= p \sin O / l (m^2+n^2-2mn \sin YOZ)^{\frac{1}{2}},$$

or $$x_0^{-1} = (m^2+n^2-2mn \sin YOZ)^{\frac{1}{2}} l / p \sin O.$$

Similarly $$y_0^{-1} = (n^2+l^2-2nl \sin ZOX)^{\frac{1}{2}} m / p \sin O,$$
$$z_0^{-1} = (l^2+m^2-2lm \sin XOY)^{\frac{1}{2}} n / p \sin O.$$

The unit factor is, therefore,

$$(m^2+n^2-2mn \sin YOZ)^{\frac{1}{2}} lx' + (n^2+l^2-2nl \sin ZOX)^{\frac{1}{2}} my'$$
$$+ (l^2+m^2-2lm \sin XOY)^{\frac{1}{2}} nz'$$

divided by $p \sin O$ or

$$p (1-\cos^2 YOZ-\cos^2 ZOX-\cos^2 XOY+2 \cos YOZ \cos ZOX \cos XOY)^{\frac{1}{2}}.$$

4. Briefly the result may be stated thus :—

If the equation to the surface be

$$F(xyz) \equiv u_n+u_{n-1}+\ldots+u_0 = 0,$$

and the plane $$v_1 \equiv ax+by+cz = 1,$$

then the equation of the section of F made by v_1 will be

$$u_n+v_1u_{n-1}+v_1^2u_{n-2}+\ldots+v_1^nu_0 = 0,$$

where the x, y, z are certain multiples of the perpendiculars $(x'y'z')$ from any point of the section on the lines in which the plane cuts the coordinate planes YOZ, ZOX, XOY; viz.,

$$x = x' (b^2+c^2-2bc \cos YOZ)^{\frac{1}{2}} \div,$$
$$y = y' (c^2+a^2-2ca \cos ZOX)^{\frac{1}{2}} \div,$$
$$z = z' (a^2+b^2-2ab \cos XOY)^{\frac{1}{2}} \div,$$

x, y, z being connected by the relation,

$$ax+by+cz=1,$$

and the divisor being the square root of

$$\Sigma a^2 \sin^2 YOZ+2\Sigma bc\,(\cos ZOX \cos XOY-\cos YOZ),$$

since $p=\sin O \div$ that same divisor.

If the axes are rectangular, then

$$x=x'\,(b^2+c^2)^{\frac{1}{2}} \div (a^2+b^2+c^2)^{\frac{1}{2}},$$
$$y=y'\,(c^2+a^2)^{\frac{1}{2}} \div (a^2+b^2+c^2)^{\frac{1}{2}},$$
$$z=z'\,(a^2+b^2)^{\frac{1}{2}} \div (a^2+b^2+c^2)^{\frac{1}{2}}.$$

5. In employing a trilinear equation, it is important to have ascertained the values of the sines and cosines of the triangle of reference. In the present case,

$$\begin{aligned}\sin A &= 2\Delta/CA\,.\,AB = 6V/p\,.\,CA\,.\,AB\\ &= OA\,.\,OB\,.\,OC \sin O/p\,.\,CA\,.\,AB\\ &= l \sin O/(n^2+l^2-2nl\cos ZOX)^{\frac{1}{2}}\,(l^2+m^2-2lm\cos XOY)^{\frac{1}{2}}\\ &= a\,\{\Sigma a^2 \sin^2 YOZ-2\Sigma bc\,(\cos YOZ-\cos ZOX \cos XOY)\}^{\frac{1}{2}} \div\\ &\qquad (c^2+a^2-2ca\cos ZOX)^{\frac{1}{2}}(a^2+b^2-2ab\cos XOY)^{\frac{1}{2}},\end{aligned}$$

according as the cutting plane is defined by

$$lx+\ldots=p \text{ or } ax+by+cz=1\,;$$

$$\cos A = (l^2+mn\cos YOZ-nl\cos ZOX-lm\cos XOY) \div$$

or

$$= (a^2+2bc\cos YOZ-ca\cos ZOX-ab\cos XOY) \div,$$

the divisors being the same as for $\sin A$.

If the plane of the section should pass through the origin, the process above followed becomes nugatory. In that case, to obtain a homogeneous equation, the equations of the surface and plane may first be transferred to any convenient point as a new origin, and the process then be carried out. Or M. Housel's plan may be adopted by using, as Cartesian axes, the lines in which the plane of section cuts two of the coordinate planes.

6. Suppose now the surface to be defined by an equation,

$$F(\alpha\beta\gamma\delta)=0,$$

among tetrahedral coordinates, the four corners of the fundamental tetrahedron being A, B, C, D; and let

$$k\alpha+l\beta+m\gamma+n\delta=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

be a plane cutting the edges AB, AC, AD in the points B', C', D'.

The coordinates of these points will be

$$\left.\begin{array}{llll} a_1 = \dfrac{l}{l-k}, & \beta_1 = \dfrac{k}{k-l}, & \gamma_1 = 0, & \delta_1 = 0 \\ a_2 = \dfrac{m}{m-k}, & \beta_2 = 0, & \gamma_2 = \dfrac{k}{k-m}, & \delta_2 = 0 \\ a_3 = \dfrac{n}{n-k}, & \beta_3 = 0, & \gamma_3 = 0, & \delta_3 = \dfrac{k}{k-n} \end{array}\right\} \quad \ldots\ldots(2).$$

If P be any point lying in the plane (1), and λ, μ, ν its areal coordinates relative to the triangle $B'C'D'$, then

$$\begin{aligned} \lambda &= PC'D' : A'C'D' \\ &= \text{vol. } PAC'D' : \text{vol. } A'AC'D' \\ &= \text{vol. } PACD : \text{vol. } A'ACD \\ &= \beta : \beta_1, \end{aligned}$$

or

$$\left.\begin{array}{l} \beta = \lambda\beta_1 = \dfrac{k\lambda}{k-l}. \\ \text{Similarly,} \quad \gamma = \dfrac{k\mu}{k-m} \\ \delta = \dfrac{k\nu}{k-n} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and, since the a of any point on the plane is equal to

$$-(l\beta + m\gamma + n\delta) \div k,$$

$$a = \frac{l\lambda}{l-k} + \frac{m\mu}{m-k} + \frac{n\nu}{n-k};$$

or, say,

$$a = l'\lambda + m'\mu + n'\nu \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

These substitutions for the tetrahedral coordinates in the equation to the surface will therefore give the equation of the section by the plane (1) in areal coordinates, the triangle of reference being that formed by the three lines in which the plane cuts the three faces of the tetrahedron meeting in the vertex A.

7. Symmetric substitutions may be obtained in lieu of the above by taking as a new triangle of reference, LMN, that formed by the three diagonals of the quadrilateral made up of the four lines in which the plane $ka + \ldots$ cuts the four faces of the tetrahedron $ABCD$; viz., the lines

$$\lambda = 0, \ \mu = 0, \ \nu = 0, \ l'\lambda + m'\mu + n'\nu = 0 \ldots\ldots\ldots\ldots(4),$$

the diagonals of which are

$$m'\mu + n'\nu = 0, \ n'\nu + l'\lambda = 0, \ l'\lambda + m'\mu = 0.$$

Let ξ, η, ζ be the areal coordinates of P relative to this new triangle; viz., $\xi = PMN : LMN$, $\eta = PNL : LMN$, $\zeta = PLM : LMN$.

Now the coordinates of L, M, N, relative to $B'C'D'$, are

$$\lambda' = -l'^{-1} \div, \quad \mu' = m'^{-1} \div, \quad \nu' = n'^{-1} \div,$$

the divisor being $-l'^{-1}+m'^{-1}+n'^{-1}$;

$$\lambda'' = l'^{-1} \div, \quad \mu'' = -m'^{-1} \div, \quad \nu'' = n'^{-1} \div,$$

the divisor being $l'^{-1}-m'^{-1}+n'^{-1}$;

$$\lambda''' = l'^{-1} \div, \quad \mu''' = m'^{-1} \div, \quad \nu''' = -n'^{-1} \div,$$

the divisor being $l'^{-1}+m'^{-1}-n'^{-1}$;

and if Δ represents the area of the triangle $B'C'D'$,

$$\text{area } PMN = \Delta \begin{vmatrix} \lambda, & \mu, & \nu \\ l'^{-1}, & -m'^{-1}, & n'^{-1} \\ l'^{-1}, & m'^{-1}, & -n'^{-1} \end{vmatrix} \div (l'^{-1}-m'^{-1}+n'^{-1})(l'^{-1}+m'^{-1}-n'^{-1})$$

$$= 2\Delta (m'\mu+n'\nu) \div l'm'n' (\ldots)(\ldots).$$

Or,

$$\text{area } PMN = 2\Delta (m'\mu+n'\nu)(-l'^{-1}+m'^{-1}+n'^{-1}) \div,$$

$$\text{area } PNL = 2\Delta (n'\nu+l'\lambda)(l'^{-1}-m'^{-1}+n'^{-1}) \div,$$

$$\text{area } PLM = 2\Delta (l'\lambda+m'\mu)(l'^{-1}+m'^{-1}-n'^{-1}) \div,$$

$$\text{area } LMN = 4\Delta \div,$$

the common divisor being

$$l'm'n' (-l'^{-1}+m'^{-1}+n'^{-1})(l'^{-1}-m'^{-1}+n'^{-1})(l'^{-1}+m'^{-1}-n'^{-1});$$

and the value of the fourth area being obtained from that of the first by writing μ', ν' for μ, ν; or, analogously from either of the other two of the first three.

Hence the equations

$$2\xi' = 2(-l'^{-1}+m'^{-1}+n'^{-1})\xi = m'\mu+n'\nu,$$

$$2\eta' = 2(\ l'^{-1}+m'^{-1}-n'^{-1})\eta = n'\nu+l'\lambda,$$

$$2\zeta' = 2(\ l'^{-1}+m'^{-1}-n'^{-1})\zeta = l'\lambda+m'\mu,$$

giving

$$l'\lambda = -\xi'+\eta'+\zeta',$$

$$m'\mu = \ \xi'-\eta'+\zeta',$$

$$n'\nu = \ \xi'+\eta'-\zeta',$$

from which, observing (3) that $l\beta \div k = -l'\lambda$, $m\gamma \div k = -m'\mu$, $n\delta \div k = n'\nu$, $-(l\beta+m\gamma+n\delta) \div k = k\alpha \div k$,

$$k\alpha = k(\xi'+\eta'+\zeta'),$$

$$l\beta = k(\xi'-\eta'-\zeta'),$$

$$m\gamma = k(-\xi'+\eta'-\zeta'),$$

$$n\delta = k(-\xi'-\eta'+\zeta').$$

Finally, substituting for l'^{-1}, m'^{-1}, n'^{-1}, their values $1-kl^{-1}$, $1-km^{-1}$, $1-kn^{-1}$,

$$-l'^{-1}+m'^{-1}+n'^{-1} = k\,(k^{-1}+l^{-1}-m^{-1}-n^{-1}),$$
$$l'^{-1}-m'^{-1}+n'^{-1} = k\,(k^{-1}-l^{-1}+m^{-1}-n^{-1}) \ldots,$$

the substitutions may be written

$$\alpha = k^{-1}\,(\xi'+\eta'+\zeta'),$$
$$\beta = l^{-1}\,(\xi'-\eta'-\zeta'),$$
$$\gamma = m^{-1}\,(-\xi'+\eta'-\zeta'),$$
$$\delta = n^{-1}\,(-\xi'-\eta'+\zeta'),$$

where now

$$\xi' = \xi \div (k^{-1}+l^{-1}-m^{-1}-n^{-1}),$$
$$\eta' = \eta \div (k^{-1}-l^{-1}+m^{-1}-n^{-1}),$$
$$\zeta' = \zeta \div (k^{-1}-l^{-1}-m^{-1}+n^{-1}).$$

8. To express the parts of the triangle of reference LMN in terms of $k \ldots n$ and the edges of the fundamental tetrahedron $ABCD$; first,

$$MN^2 = (\mu''-\mu''')(\nu'''-\nu'')\,C'D'^2+(\nu''-\nu''')(\lambda'''-\lambda'')\,D'B'^2 + (\lambda''-\lambda''')(\mu'''-\mu'')\,B'C'^2,$$

$$= 2l'^{-2}\{m'^{-1}n'^{-1}C'D'^2+n'^{-1}(-m'^{-1}+n'^{-1})\,D'B'^2 + m'^{-1}(m'^{-1}-n'^{-1})\,B'C'^2\} \div (l'^{-1}-m'^{-1}+n'^{-1})^2\,(l'^{-1}+m'^{-1}-n'^{-1})^2$$

$$= 2\,(l-k)^2\{mn\,(m-k)(n-k)\,C'D'^2+(n-k)(n-m)\,kmD'B'^2 + (m-k)(m-n)\,knB'C'^2\} \div k^2l^2m^2n^2\,(k^{-1}-l^{-1}+m^{-1}-n^{-1})^2\,(k^{-1}-l^{-1}-m^{-1}+n^{-1})^2.$$

Representing the edges of the tetrahedron by

$$CD = a,\ DB = b,\ BC = c,\ AB = a',\ AC = b',\ AD = c',$$

$$C'D'^2 = \gamma_2\delta_3 a^2+(a_3-a_2)\,\gamma_2 b'^2+(a_2-a_3)\,\delta_3 c'^2$$
$$= k^2\,(m-k)^{-1}\,(n-k)^{-1}\,a^2+k^2\,(m-n)(m-k)^{-2}\,(n-k)^{-1}\,b'^2 + k^2\,(n-m)(n-k)^{-2}\,(m-k)^{-1}\,c'^2;$$

$$D'B'^2 = k^2\,(n-k)^{-1}\,(l-k)^{-1}\,b^2+k^2\,(n-l)(n-k)^{-2}\,(l-k)^{-1}\,c'^2 + k^2\,(l-n)(l-k)^{-2}\,(n-k)^{-1}\,a'^2;$$

$$B'C'^2 = k^2\,(l-k)^{-1}\,(m-k)^{-1}\,c^2+k^2\,(l-m)(l-k)^{-2}\,(m-k)^{-1}\,a'^2 + k^2\,(m-l)(m-k)^{-2}\,(l-k)^{-1}\,b'^2.$$

Substituting these values in the expression above for MN^2,

$$MN^2 = 2\,\{mn\,(l-k)^2\,a^2-kl\,(m-n)^2\,a'^2 + (l-k)(n-m)(kmb^2-knc^2-nlb'^2+lmc'^2)\} \div k^2l^2m^2n^2\,(k^{-1}-l^{-1}+m^{-1}-n^{-1})^2\,(k^{-1}-l^{-1}-m^{-1}+n^{-1})^2.$$

The terms within the { } have a remarkable significance; for it

may readily be verified that they form the expression for the square of the line (a_1), joining the points in which the plane $k\alpha+\ldots$ cuts the opposite edges AB, CD of the tetrahedron $ABCD$, multiplied by the factor $(k-l)^2\,(m-n)^2$;

thus $MN = \sqrt{2}\,(k-l)(m-n)\,a_1 \div klmn\,(k^{-1}-\ldots)(k^{-1}-\ldots)$;

similarly $NL = \sqrt{2}\,(k-m)(n-l)\,b_1$
$\div klmn\,(k^{-1}-l^{-1}-m^{-1}+n^{-1})(k^{-1}-l^{-1}-m^{-1}-n^{-1})$,

$LM = \sqrt{2}\,(k-n)(l-m)\,c_1$
$\div klmn\,(k^{-1}+l^{-1}-m^{-1}-n^{-1})(k^{-1}-l^{-1}+m^{-1}-n^{-1})$,

b_1, c_1 being the lengths of the connectors of the points in which the plane $k\alpha+\ldots$ meets AC, BD and AD, BC respectively.

The area of the triangle LMN has been expressed in terms of that of the triangle $B'C'D'$. If V represents the volume of the tetrahedron $ABCD$, V' that of the tetrahedron $AB'C'D'$, and p the length of the perpendicular from A on the plane $k\alpha+\ldots$; then

$$p\,.\,B'C'D' = 3V' = 3AB'\,.\,AC'\,.\,AD'\,.\,V \div AB\,.\,AC\,.\,AD.$$

Now $p = 3V\,.\,k \div (k^2A^2+\ldots-2mnCD\cos\widehat{CD})^{\frac{1}{2}}$,

where $A\ldots D$ stand for the areas of the faces opposite the corners $A\ldots D$; and

$$AB' : AB = k : k-l,\ AC' : AC = k : k-m,\ AD' : AD = k : k-n,$$

so that the area

$$B'C'D' = k^2\,(k^2A^2+\ldots-2mn\,CD\cos\widehat{CD})^{\frac{1}{2}} \div (k-l)(k-m)(k-n),$$

and it was shown that

$$LMN = 4B'C'D' \div l'm'n'\,(-l'^{-1}+\ldots)(l'^{-1}+\ldots)(l'^{-1}+m'^{-1}-n'^{-1})$$
$$= 4\,(l-k)(m-k)(n-k)\,B'C'D' \div k^3lmn\,(k^{-1}+l^{-1}-m^{-1}-n^{-1})(\ldots)(\ldots).$$

Hence, finally,

$$LMN = 4\,(k^2A^2+\ldots-2mn\,CD\cos\widehat{CD})^{\frac{1}{2}} \div$$
$$klmn\,(k^{-1}+l^{-1}-m^{-1}-n^{-1})(k^{-1}-l^{-1}+m^{-1}-n^{-1})(k^{-1}-l^{-1}-m^{-1}+n^{-1}).$$

9. If the original equations of the surface and plane had been given in quadriplanar coordinates, then everywhere, in what precedes, $k\alpha_0$, $l\beta_0$, $m\gamma_0$, $n\delta_0$ should be written for $k\ldots\delta$ respectively, $\alpha_0\ldots\delta_0$ being the perpendiculars from the four vertices of the tetrahedron of reference on the opposite faces; viz.,

$$\alpha_0 = 3V/A,\quad \beta_0 = 3V/B,\quad \gamma_0 = 3V/C,\quad \delta_0 = 3V/D.$$

And to obtain the symmetrical trilinear equation of the section for ξ, η, ζ, in the values of $\alpha\ldots\delta$, must, of course, be written $\xi/x_0\ldots$, $x_0y_0z_0$, being the perpendiculars from $L\ldots$ on the opposite

sides MN ... of the triangle LMN; viz.,

$$x_0 = 2LMN/MN = 24V(k^2+\ldots-2mn\cos\widehat{CD})^{\frac{1}{2}}$$
$$\times[(ka_0)^{-1}+(l\beta_0)^{-1}-(m\gamma_0)^{-1}-(n\delta_0)^{-1}]\div\sqrt{2}\,(ka_0-l\beta_0)(m\gamma_0-n\delta_0)\,a_1;$$
$$y_0 = 24V(k^2+\ldots)^{\frac{1}{2}}[(ka_0)^{-1}-(l\beta_0)^{-1}+(m\gamma_0)^{-1}-(n\delta_0)^{-1}]$$
$$\div\sqrt{2}\,(ka_0-m\gamma_0)(n\delta_0-l\beta_0)\,b_1,$$
$$z_0 = 24V(k^2+\ldots)^{\frac{1}{2}}[(ka_0)^{-1}-(l\beta_0)^{-1}-(m\gamma_0)^{-1}+(n\delta_0)^{-1}]$$
$$\div\sqrt{2}\,(ka_0-n\delta_0)(l\beta_0-m\gamma_0)\,c_1.$$

Motion of a Network of Particles with some Analogies to Conjugate Functions. *By* Dr. E. J. Routh.

[*Read May 8th*, 1884.]

Summary.—In this paper we propose to discuss briefly the motion of certain networks with material particles placed at the intersection of the threads. We begin with a network in which the openings between the threads are rectangular. Expressing the equations of motion by a single equation of differences and writing down the solution, we find that there are at least two distinct kinds of motion. Either of these might be produced by a forced agitation at a boundary. The distinction between the two types of motion depends on the period of the forced oscillation. If this period exceed a certain limit, waves will travel over the network. If the period fall short of this limit, the distant parts of the network will be permanently at rest. There are also other differences between the two kinds of motion. When the network reduces to a membrane, the second kind of motion is found not to exist.

When the network is equally stretched in all directions, the wave motions will travel quickest when the front is along the diagonals of the openings, and slowest when the fronts are along the threads.

Passing next to the case of a network in which the openings form quadrilaterals, we find that there is a class of such networks which move exactly as if the openings were rectangular. A criterion is given to distinguish such networks and also a method of constructing them. Thus the periods of oscillation of any rectangular network being known, we can immediately state the periods of oscillation of the corresponding quadrilateral network.

By noticing the meaning of this transformation when the network is so fine as to become a membrane, we find it is analogous to that of conjugate functions. Thus the transformation here effected for the

equation of finite differences corresponds (to a certain degree) to those effected for the partial differential equation by the use of conjugate functions. Examining this transformation more minutely, several theorems are proved to be true which reduce in the limit to some well-known theorems in conjugate functions.

Finally, two theorems on the motion of a wave over a quadrilateral network are alluded to.

Rectangular Networks.

1. *Equations of Motion.*—Let columns of threads in one plane be cut at right angles by rows of threads. Let a particle of mass m be placed at each intersection and attached to the two threads. Let the interval between two adjacent columns be l, and the interval between two adjacent rows be l'. Let the tensions of the rows and columns be respectively T and T'. Let the particles vibrate perpendicularly to the plane of the threads, and let the whole system be removed from the action of gravity.

Let w, or more properly w_{hk}, be the displacement of the particle in the h^{th} column and k^{th} row. This particle is acted on by four threads. Hence its equation of motion is

$$m\frac{d^2w}{dt^2}=\frac{T}{l}(w_{h+1}-w)-\frac{T}{l}(w-w_{h-1})+\frac{T'}{l'}(w_{k+1}-w)-\frac{T'}{l'}(w-w_{k-1}).$$

Putting $\frac{T}{ml}=c^2$, $\frac{T'}{ml'}=c'^2$, this equation becomes

$$\frac{d^2w}{dt^2}=c^2(w_{h+1}-2w+w_{h-1})+c'^2(w_{k+1}-2w+w_{k-1}).$$

Using the notation of the Calculus of Finite Differences, this may be written

$$\frac{d^2w_{hk}}{dt^2}=c^2\Delta^2w_{h-1,k}+c'^2\Delta'^2w_{h,k-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where Δ operates on h and Δ' on k.

2. We may notice that, if $cl=c'l'$, we have $Tl=T'l'$; so that the tensions are proportional to the sides of the openings in the network across which they act. In the limit when the network is infinitely close, this would make the tensions across all elementary lines to be the same when referred (as usual) to a unit of length. The membrane may then be said to be equally stretched throughout.

3. The equation (1) represents the equation of motion of every particle except those at the boundary of the network. In order to include these, it is necessary to make some suppositions about the points of attachment of the strings, so that the terms depending on the non-existent particles may be equal to zero. Thus, if it be given that the

boundary is constrained to move in a given manner, we suppose that this boundary does not intersect the strings except at points at which particles might be placed, and which are, therefore, defined by integer values of h and k. If the actual boundary does not satisfy this condition, we must replace it by a slightly zig-zag boundary which passes through the neighbouring particles.

4. *Different forms of the Solution.*—To solve this equation of mixed differences, we put $w = A a^h b^k \sin pt$ .....(2).

There is, of course, another similar term with $\cos pt$ instead of $\sin pt$, which we need not consider at present. Substituting in the equation of differences, we obtain

$$-p^2 = c^2\left(a-2+\frac{1}{a}\right)+c'^2\left(b-2+\frac{1}{b}\right) \quad \text{.....(3).}$$

This last equation shows that, if a and b be both real, one or both must be negative unless p is zero and a, b each equal to unity. For each value of a and p, we obtain two corresponding values of b which are reciprocals of each other. The solution may therefore be written

$$w = \Sigma\{a^h(Ab^k+Bb^{-k})+a^{-h}(A'b^k+B'b^{-k})\}\sin pt \quad \text{.....(4),}$$

where p, a, and b are connected by the equation (3). If a be real and b imaginary with a modulus unity, we have

$$w = \Sigma\{a^h(A\cos 2k\phi+B\sin 2k\phi)+a^{-h}(A'\cos 2k\phi+B'\sin 2k\phi)\}\sin pt \quad \text{.....(5),}$$

where p, a, and b are connected by the equation

$$p^2 = -c^2\left(a-2+\frac{1}{a}\right)+c'^2(2\sin\phi)^2 \quad \text{.....(6).}$$

The imaginary values of a and the real values of b give similar terms.

If both a and b be imaginary, we write $a = r(\cos 2\theta+\sin 2\theta\sqrt{-1})$, $b = \rho(\cos 2\phi+\sin 2\phi\sqrt{-1})$, with a relation between r, ρ, θ, ϕ to make p^2 real. The result is easily written down, but, as it is not required in this paper, we restrict ourselves to the case in which r and ρ are both unity. The expression for w may then be resolved into terms of the form $w = \Sigma P \sin(pt-2h\theta-2k\phi)$ .....(7),

where $p^2 = c^2(2\sin\theta)^2+c'^2(2\sin\phi)^2$ .....(8).

5. If the circumstances of the problem allow $b = \pm 1$ to be a possible value of b, the solution (4) appears to lose one of its constants. Correcting this in the usual way, we must add to this solution the term $(\pm 1)^k\{a^h(A+Bk)+a^{-h}(A'+B'k)\}\sin pt$ .....(9).

The case in which $a = \pm 1$ is a possible value of a may be treated in the same way.

In the cases in which $p = 0$, $p = 2c$, or $p = 2c'$ are possible values of p, the solutions require further additions. In these cases, the quadratics (3) are satisfied by equal values of a or equal values of b, and thus constants are lost. When $p = 0$, we add to (4)

$$A + Bh + Ck + Dhk \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10);$$

when $p = 2c$, we add the term

$$(A + Bh + Ck + Dhk)\,(-1)^h \,.\, \sin 2ct \ldots\ldots\ldots\ldots\ldots\ldots (11).$$

6. Looking at these results, we see two distinct forms of solution. In the solution (4), the abscissæ h and k enter as exponents of real quantities; in the solution (7), they so enter as to constitute a wave motion. Thus two distinct forms of motion may be propagated in such a network. We now propose to make a few remarks on these two solutions, and also to consider whether they can both exist as the network approaches nearer to the state of a membrane.

7. *Motions of Waves.*—Consider the wave motion given by

$$\left.\begin{aligned} w &= P \sin (pt - 2h\theta - 2k\phi) \\ p^2 &= c^2 (2 \sin \theta)^2 + c'^2 (2 \sin \phi)^2 \end{aligned}\right\}.$$

The front of the wave is given by $h\theta + k\phi = A$, where A is any constant. Let ξ be the length of the perpendicular from the origin on this front. Let α be the angle this perpendicular makes with the direction of the rows. Then, since the intercepts of the front on the axes are Al/θ and Al'/ϕ, we have

$$\frac{A^2}{\xi^2} = \frac{\theta^2}{l^2} + \frac{\phi^2}{l'^2}, \qquad \tan \alpha = \frac{l}{l'}\,\frac{\phi}{\theta}.$$

The wave motion may therefore be represented by

$$w = P \sin \left(pt - 2\xi \sqrt{\frac{\theta^2}{l^2} + \frac{\phi^2}{l'^2}} \right).$$

Let λ be the length of this wave and v the velocity with which it travels, then $\qquad \theta = \dfrac{\pi l}{\lambda} \cos \alpha, \qquad \phi = \dfrac{\pi l'}{\lambda} \sin \alpha,$

$$\left(\frac{\pi}{\lambda}\right)^2 v^2 = c^2 \sin^2 \left(\frac{\pi l}{\lambda} \cos \alpha \right) + c'^2 \sin^2 \left(\frac{\pi l'}{\lambda} \sin \alpha \right).$$

8. It appears from these results that *waves of the same length travel in different directions with different velocities.*

To find the directions in which a wave of given length will travel quickest or slowest, we make $dv/d\alpha = 0$. If the network be equally stretched throughout, so that $cl = c'l' = \gamma$, we easily find that *there are two directions in which the wave will travel with the greatest velocity, and in these cases the fronts are the diagonals of the openings of the network.* Thus these directions are independent of the length of

the wave. If D be the length of either diagonal, the velocity of either of these waves is $= \gamma \dfrac{\sin \frac{\pi}{\lambda} \frac{ll'}{D}}{\frac{\pi}{\lambda} \frac{ll'}{D}}$.

There are two directions in which the waves will travel with a minimum velocity, and in these two cases the fronts are along the threads. The two velocities are not equal, being respectively

$$\gamma \frac{\sin \frac{\pi}{\lambda} l}{\frac{\pi}{\lambda} l} \text{ and } \gamma \frac{\sin \frac{\pi}{\lambda} l'}{\frac{\pi}{\lambda} l'}.$$

9. If the waves be very long compared with the dimensions of the openings of the network, we have, by Art. 7,

$$v^2 = c^2 l^2 \cos^2 \alpha + c'^2 l'^2 \sin^2 \alpha,$$

so that waves of all lengths will travel with the same velocity. If $cl = c'l'$, the velocity will be independent of the direction also.

10. *Two types of motion.*—Let us now suppose the network to be bounded on one side only by the straight line $h+k=0$ (or in Cartesian coordinates $l'x+ly=0$) and infinitely produced in all other directions. Let the particles in this straight line be attached to a rod, and let this rod be agitated so that its motion is given by

$$w = P \sin pt.$$

We see at once that waves will travel from this rod in the direction of maximum velocity. The motion is given by

$$w = P \sin \{ pt - 2\theta (h+k) \},$$
$$p^2 = (c^2+c'^2) (2 \sin \theta)^2.$$

The second equation shows that there is a superior limit to the value of p. If the period of the agitating rod be greater than that determined by $p^2 = 4 (c^2+c'^2)$, these equations represent the motion. But, if the period of the forced oscillation be less than this limit, the value of θ is imaginary. In this case the motion takes another type. This is easily obtained from equation (4), Art. 4. We have

$$w = P (-a)^{h+k} \sin pt,$$
$$p^2 = (c^2+c'^2) \left(a+2+\frac{1}{a} \right),$$

where a is greater than unity.

This is a perfectly distinct type of motion from the former. There is no wave travelling over the network. The agitation of the particles remote from the rod is very slight. Tracing the motion more minutely, we see that the particles on alternate diagonals are always on opposite sides of the plane of the network.

11. That there should be two kinds of motion depending on the period of the agitating rod, was perhaps to be expected beforehand. The rod sets the next line of particles in motion by means of the resolved tensions of the strings. Thus a certain time is required for each line of particles to communicate a velocity equal to its own to the next line; and this time, with given tensions and displacements, is longer the greater the length of the connecting strings. If then the period of the forced vibration be less than this time, it is clear that the extent of oscillation of each line of particles must be greater than that of the next following line. Thus the motion could not be that of a wave which travels unchanged over the network.

12. It is interesting to determine what this limiting time, viz. $\pi / (c^2+c'^2)^{\frac{1}{2}}$, becomes when the network is so fine that it may be regarded as a membrane. Let U and U' be the principal tensions referred to a unit of length, so that $T = Ul'$, $T' = U'l$. Let ρ be the quantity of matter distributed over a unit of area, so that $m = \rho ll'$. Then, by Art. 1, we easily see that $c^2l^2 = U/\rho$, $c'^2l'^2 = U'/\rho$. These must be finite, because they are the squares of the velocities of a wave in a membrane. Hence, as the membrane gets finer, c and c' both increase without limit. Thus the limiting time finally vanishes. It follows that the second type of motion cannot exist in any membrane if it be so constituted that U/ρ and U'/ρ are finite.

Quadrilateral Networks.

13. Let us now consider the case in which the threads joining the particles form quadrilateral instead of rectangular openings. To bring these into order, we must regard them as arranged in rows and columns as before, though these are no longer straight lines. We run two sets of curves through the particles, and call these our rows and columns. Let w_{hk} be the displacement of the particle in the h^{th} column and k^{th} row. Let T_{hk} be the tension of the thread joining the particles (h, k) $(h+1, k)$, and T'_{hk} the tension of that joining (hk), $(h, k+1)$. Similarly let l_{hk}, l'_{hk} be the lengths of these two threads. The equation of motion of the particle at (hk) is

$$m\frac{d^2}{dt^2}w_{hk} = T_{hk}\frac{w_{h-1}-w}{l_{hk}} - T_{h-1,k}\frac{w-w_{h-1}}{l_{h-1,k}} + T'_{hk}\frac{w_{k+1}-w}{l'_{hk}} - T'_{h,k-1}\frac{w-w_{k-1}}{l'_{h,k-1}} \quad \ldots\ldots\ldots\ldots(1).$$

If the quadrilateral network be so arranged that

$$\frac{T_{hk}}{l_{hk}} = \frac{T_{h-1,k}}{l_{h-1,k}} = mc^2, \quad \frac{T'_{hk}}{l'_{hk}} = \frac{T'_{h-1,k}}{l'_{h-1,k}} = mc'^2 \quad \ldots\ldots\ldots\ldots(2),$$

this equation reduces to

$$\frac{d^2}{dt^2}w_{hk} = c^2\Delta^2 w_{h-1,k} + c'^2\Delta'^2 w_{h,k-1} \quad \ldots\ldots\ldots\ldots(3),$$

where Δ operates on h and Δ' on k.

This is exactly the same equation as that which determines the motion of a rectangular network (Art. 1). We therefore infer that the *motions of some quadrilateral networks are exactly the same as the motion when the particles are arranged in the form of a rectangular network.* The initial and boundary conditions are, of course, supposed to correspond also.

14. To determine what quadrilateral networks possess this property we must examine equations (2). These express the conditions that the *tension of every string in every row should be proportional to its length.* A similar relation holds for the tensions of the string in every column.

15. Besides these conditions, we have those derived from the equilibrium of the network when undisturbed. The four tensions on any particle must be in equilibrium. But the tensions are proportional to the lengths of the strings along which they act, the ratios for the rows and columns being mc^2 and mc'^2. If $c^2 = c'^2$, we see at once, by Pappus' theorem, that *each particle is the centre of gravity of the four adjacent particles to which it is connected by strings.* If c^2 is not equal to c'^2, the condition is equally evident, though slightly more complicated. We must now multiply the masses of the adjacent particles by c^2 or c'^2, according as they lie in the same row or column. With this change, each particle must be the centre of gravity of the four adjacent particles.

16. The motions of all quadrilateral networks which possess this property follow at once from that of a rectangular network. The equations of motion, when expressed in terms of h and k, are exactly the same.

17. It is clear that we may geometrically construct an infinite number of quadrilateral networks which possess this property. Starting with any four particles, we may arrange the other particles in an infinite number of ways by placing them so that each particle already placed is the centre of gravity of the four adjacent particles.

[Let us suppose the particles thus arranged, so that each is at the centre of gravity of the four adjacent particles. Let us also suppose that the network has two boundaries defined (as explained in Art. 3) by $h = 0$, $k = 0$ respectively, and let the strings attached to these boundaries be pulled so that their tensions are all proportional to their lengths. By considering the conditions of equilibrium of the several particles which lie in the column and row next to these boundaries (*i.e.*, the column and row defined by $h = 1$, $k = 1$ respectively), we can prove that the tensions of the strings which join these to the particles in the column and row defined by $h = 2$, $k = 2$ re-

spectively are also proportional to their lengths. Proceeding in this manner, we may show that the tension of every string in the interior of the network is proportional to its length. Thus the conditions of Art. 13 are satisfied.]

18. We may construct the network analytically by expressing the coordinates of the several particles. Let x_{hk}, y_{hk} be the Cartesian coordinates of the particle in the h^{th} column and k^{th} row referred to any rectangular axes. Then, by a common formula for the centre of gravity, $c^2(x_{h+1}-2x+x_{h-1})+c'^2(x_{k+1}-2x+x_{k-1})=0$,

which may (with the corresponding equation for y) be written

$$\left.\begin{aligned} c^2\Delta^2 x_{h-1,k}+c'^2\Delta'^2 x_{h,k-1}&=0\\ c^2\Delta^2 y_{h-1,k}+c'^2\Delta'^2 y_{h,k-1}&=0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where Δ operates on h and Δ' on k.

19. To solve these equations, we put $x=Aa^hb^k$; we thus have

$$c^2\left(a-2+\frac{1}{a}\right)+c'^2\left(b-2+\frac{1}{b}\right)=0 \quad \ldots\ldots\ldots\ldots\ldots(5).$$

Reasoning exactly as in Art. 4, we have

$$x=\Sigma\{a^h(Ab^k+Bb^{-k})+a^{-h}(A'b^k+B'b^{-k})\} \quad \ldots\ldots\ldots\ldots(6).$$

It is clear that (except $a=1$, $b=1$) a and b cannot be both positive quantities. When a and b are both unity, we have, as in Art. 5,

$$x=A+Bh+Ck+Dhk \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

If a and b be imaginary, let us put $a=e^{2\alpha}$ and $b=e^{2\beta\sqrt{-1}}$. We then have, from (5),

$$\frac{c}{c'}\tfrac{1}{2}(e^{\alpha}-e^{-\alpha})=\pm\sin\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where α and β are real or imaginary.

20. Since x and y are to be found in terms of h and k by the same equation (4), it will be more convenient to write the solution in the form $x+y\sqrt{-1}=\Sigma Ae^{2\alpha h+2\beta k\sqrt{-1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$;

then x and y are any sets of terms selected out of the real and imaginary parts respectively.

21. Example 1.—Let a network with rectangular openings be inserted into a rectangular framework, so that the boundaries are fixed. We have then $w_{hk}=0$ when $h=0$ and $h=n+1$ for all values of k, also $w_{hk}=0$ when $k=0$ and $k=n'+1$ for all values of h. By adding several waves of the form (7) in Art. 4, we easily find

$$w=P\sin 2h\theta\sin 2k\phi\sin(pt+\gamma) \quad \ldots\ldots\ldots\ldots\ldots(\text{I.}),$$

$$\left.\begin{aligned} p^2&=c^2(2\sin\theta)^2+c'^2(2\sin\phi)^2\\ 2(n+1)\theta&=i\pi,\ 2(n'+1)\phi=i'\pi\end{aligned}\right\}.$$

Let us transform this result by writing

$$x = Ae^{2ah}\cos 2\beta k, \quad y = Ae^{2ah}\sin 2\beta k.$$

Eliminating h and k in turn, we have

$$x^2+y^2 = A^2e^{4ah}, \quad \frac{y}{x} = \tan 2\beta k.$$

Thus the columns have been turned into circles, and the rows into radii vectores. The result may be stated thus. Draw concentric circles whose radii form the geometrical progression A, Ae^{2a}, Ae^{4a}, &c. Draw radii vectores so that the angle between each consecutive two is 2β. Let the circles defined by $h = 0$, $h = n+1$, and the radii vectores defined by $k = 0$, $k = n'+1$, be all four fixed. Then the motion of this network is given by equations (I.).

22. Example 2. — Another example is found by taking $x = h$, $y = hk$. In this case, the columns are unchanged, but the rows become radii vectores, their equation being $y = kx$. Thus the equations (I.) of the last example give the oscillations of a network whose boundary is a right-angled triangle, the rows passing through one of the acute angles, and the columns being parallel to the opposite side.

23. In constructing the networks described in these two examples, we must, of course, attend to the necessary conditions at the boundaries. The strings are to be attached to the fixed boundaries, and each string is to be so tightened that the tensions at the boundaries satisfy the conditions given in Art. 14. Then these conditions are satisfied throughout the interior of the membrane.

24. *Conjugate Functions.*—Returning to the general formulæ given in Art. 19, let us suppose the openings in the network to be very small. In this case, x and y must receive very small increments when h and k increase by unity. We must therefore have α and β both small. It follows, from equation (8), that

$$\frac{c}{c'}\alpha = \pm\beta \qquad \text{.....(10).}$$

Substituting in (9), we find

$$x+y\sqrt{-1} = \Sigma Ae^{2a\left(h\pm\frac{c}{c'}k\sqrt{-1}\right)}.$$

But, since A and a are arbitrary, this gives

$$x+y\sqrt{-1} = f\left(h+\frac{c}{c'}k\sqrt{-1}\right) \qquad \text{.....(11).}$$

Thus x and y are conjugate functions of h and kc/c'. *The equations* (9) *of Art.* 20 *may therefore be regarded as an extension of conjugate functions to particles at finite distances.*

The transformation used in Ex. 1, Art. 21, evidently corresponds

to the common transformation in conjugate functions

$$\left.\begin{aligned} x+y\sqrt{-1} &= e^{h+k\sqrt{-1}} \\ \text{or} \qquad h+k\sqrt{-1} &= \log(x+y\sqrt{-1}) \end{aligned}\right\}.$$

25. In a paper published in the *Proceedings* of the Mathematical Society, 1882, the author gave a method of finding the periods of oscillation of certain heterogeneous membranes when the periods of the corresponding homogeneous membranes are known. The method now described is analogous to the one there given. When the openings of the network are rectangles and the masses at the corners equal, the analogy to a homogeneous membrane is evident. When the openings of the network are changed into quadrilaterals, the areas of these are not equal, and the limiting case of such a network is a heterogeneous membrane. But the same rule holds in both cases; viz., *the total oscillating mass is the same over all corresponding areas in the homogeneous and heterogeneous membrane or network, but is differently distributed.*

26. *Analogies to Conjugate Functions.*—We may notice some further analogies between the values of x and y, given by equation (9) of Art. 20, and the conjugate functions given by (11). In equation (9) the quantities h and k are restricted to be *integers*, but, if we give h and k all possible real values, we are drawing continuous curves which run through all the particles. These curves may be called the rows and columns according as we make k or h constant. With this understanding, we have

$$x_{hk}+y_{hk}\sqrt{-1} = \Sigma A e^{2(\alpha h+\beta k\sqrt{-1})} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

We easily find, by substitution, if we remember the relation (8) of Art. 19 between α and β, that

$$\begin{aligned} &(x_{h+\frac{1}{2}}+y_{h+\frac{1}{2}}\sqrt{-1})-(x_{h-\frac{1}{2}}+y_{h-\frac{1}{2}}\sqrt{-1}) \\ &\quad = \pm\frac{c'\sqrt{-1}}{c}\{(x_{k+\frac{1}{2}}+y_{k+\frac{1}{2}}\sqrt{-1})-(x_{k-\frac{1}{2}}+y_{k-\frac{1}{2}}\sqrt{-1})\}. \end{aligned}$$

Equating the real and imaginary parts of this equation, we have

$$\left.\begin{aligned} x_{h+\frac{1}{2}}-x_{h-\frac{1}{2}} &= \mp\frac{c'}{c}(y_{k+\frac{1}{2}}-y_{k-\frac{1}{2}}) \\ x_{k+\frac{1}{2}}-x_{k-\frac{1}{2}} &= \pm\frac{c}{c'}(y_{h+\frac{1}{2}}-y_{h-\frac{1}{2}}) \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots (12).$$

These equations, written in the language of the Calculus of Finite Differences, become

$$\left.\begin{aligned} \Delta x_{h-\frac{1}{2},k} &= \mp\frac{c'}{c}\Delta' y_{h,k-\frac{1}{2}} \\ \Delta y_{h-\frac{1}{2},k} &= \pm\frac{c'}{c}\Delta' x_{h,k-\frac{1}{2}} \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (13),$$

and in this form (especially when c' is put equal to c) their analogy to conjugate functions is evident.

27. We may express these results in a geometrical form. Consider any particle (hk); take two points P, P' on the same row, so that the abscissæ of P and P' are the arithmetic means of the abscissæ of (hk), $(h+1, k)$ and $(h-1, k)$, (hk) respectively. Next, take two points Q, Q' on the same column on which (hk) lies, so that the ordinates of Q and Q' are the arithmetic means of the ordinates of (hk), $(h, k+1)$ and $(h, k-1)$, (hk) respectively. The directions of these abscissæ and ordinates in space are arbitrary, but at right angles. Then we easily show, from (12), *that PP' and QQ' are at right angles, and their lengths are in the ratio of c' to c.* Thus, when the network is so stretched that $c = c'$, PP' and QQ' are two equal lines at right angles.

When the network becomes a membrane, the lengths PP', QQ' become elementary arcs of the conjugate curves. The analogy of these geometrical results to those of conjugate curves is therefore obvious.

28. *Waves on quadrilateral networks.*—We may now consider the motion of the wave whose front runs along the diagonals of the openings. This is the wave whose motion on a rectangular network has been considered in Art. 8. In that case the front was necessarily a straight line, but in a quadrilateral network the front may have any form. The motion is, however, still represented by the equation

$$w = P \sin [\{(c^2+c'^2)^{\frac{1}{2}} 2 \sin \theta\} t - 2\theta (h+k)],$$

as explained in Art. 7.

In the time $$\tau = \frac{\theta}{\sin \theta} \frac{1}{(c^2+c'^2)^{\frac{1}{2}}},$$

the wave front has advanced from one line of diagonals to the next. Thus the wave does not travel equally quickly in all directions, and its front does not retain the same shape. But there is this peculiarity in its motion, that, *in equal times (multiples of τ), the same front passes over equal quantities of matter*, instead of describing equal spaces.

29. We may also notice another peculiarity, that the *limiting time of oscillation*, viz., $\frac{\pi}{(c^2+c'^2)^{\frac{1}{2}}}$ (see Art. 12), *which separates the two kinds of motion which a network can assume is the same at all parts of these membranes.* Thus the wave motion just described cannot change into the other type of motion as it travels over the network.

Sopra una Trasformazione birazionale, del sesto grado, dello spazio a tre dimensioni, la cui inversa è del quinto grado. By Professor Cremona, F.R.S., LL.D. (Edin.)

[*Read May 8th*, 1884.]

1. La breve comunicazione che ho l'onore di fare alla London Mathematical Society è sostanzialmente una illustrazione de' metodi di trasformazione geometrica dello spazio a tre dimensioni, che io cominciai a pubblicare già da molti anni, nel *Journal für die reine und angewandte Mathematik*, nei *Rendiconti dell' Istituto Lombardo*, nelle *Nachrichten della R. Società di Göttingen*, nelle *Memorie dell' Accademia di Bologna*, negli *Annali di Matematica*, etc.: i quali metodi sono fondati sulla teoria delle trasformazioni delle figure piane, da me dato nel 1865 (*Memorie dell' Accademia di Bologna*).

2. Se Ψ è una superficie di 3° ordine, dotata di un punto doppio (conico) O, e se per cinque delle sei rette di Ψ uscenti da O si fa passare un cono generale di 4° ordine, la rimanente intersezione sarà una curva di 7° ordine Γ_7, per la quale O è un punto triplo. Si sa che questa curva Γ_7 (per la quale passa una *rete*, ossia un sistema lineare ∞^2 di superficie analoghe a Ψ) può essere presa come curva doppia di infinite superficie di 6° ordine, aventi anch' esse in O un punto triplo. È anche noto* che una siffatta superficie può essere rappresentata in un piano ϕ per modo che le imagini delle due sezioni piane siano curve di 4° ordine aventi in comune dieci punti (fondamentali) semplici $m_1, m_2, \ldots m_{10}$. Indicherò tali curve col simbolo $c_4 \equiv m_1 m_2 \ldots m_{10} \equiv m$. L'imagine di Γ_7 è una curva $c_{11} \equiv m^3$, ossia d'ordine 11 e passante tre volte per ciascun punto m. Un' altra superficie dello stesso ordine e dotata della stessa singolarità sega ulteriormente la prima in una curva di 8° ordine, la cui imagine è una conica descritta arbitrariamente nel piano ϕ. Supponendo che la conica sia fatta passare pei tre punti m_8, m_9, m_{10}, la curva d'intersezione si spezza in tre rette $RR'R''$ trisecanti di Γ_7 ed in una quintica (razionale) C_5, che incontra le rette R ciascuna una volta e la Γ_7 tredici volte. Chiamerò Φ_6 o Φ la superficie rappresentata in ϕ e Φ', Φ''... le altre superficie analoghe, seganti Φ nelle stesse tre rette R (oltre la curva doppia Γ_7); tutte le Φ formano un sistema *omaloidale* (ossia tale che tre punti arbitrari dello spazio individuano una Φ, e tre Φ si segano in un solo punto, all' infuori delle linee *fondamentali* Γ_7, R_1, R_2, R_3). Tutte le quintiche analoghe C_5 sono in numero $(4.5-3-13=)$ 4 volte infinito, e due di

* Caporali nei *Collectanea Mathematica in memoriam D. Chelini*, p. 169. Cfr. una mia comunicazione alla R. Irish Academy, 28 aprile 1884.

esse, se poste in una stessa superficie Φ, si segano in un solo punto. Tutto ciò appare manifesto dalla rappresentazione ϕ.

3. Per lo scopo della presente memoria, conviene di trasformare il piano ϕ mediante una trasformazione quadratica i cui punti fondamentali siano m_8, m_9, m_{10}. Allora le imagini delle sezioni piane di Φ saranno curve di 5° ordine aventi in comune tre punti doppi a_1, a_2, a_3 e sette punti semplici b_1, $b_2 \ldots b_7$; le rette a_2a_3, a_3a_1, a_1a_2 rappresenteranno le rette R, R', R''; e la curva Γ_7 sarà rappresentata da una curva del 13° ordine, $c_{13} \equiv a^5b^3$, passante cinque volte per ciascun punto a e tre volte per ciascun punto b. Le imagini delle curve C_5 saranno ora le rette del piano ϕ.

4. Ora io posso far corrispondere univocamente i punti del dato spazio S a quelli di un altro spazio s, in modo che le superficie Φ e le curve C_5 corrispondano ai piani ϕ ed alle rette di s. Si tratta di trovare quali superficie e quali curve di s corrisponderanno ai piani e alle rette di S. Di queste superficie e curve sappiamo soltanto finora che le prime sono di 5° ordine e le seconde del 6°: come risulta dal segare in S un piano con una C_5 e una retta con una Φ.

5. La Jacobiana delle Φ è una superficie d'ordine $4(6-1) = 20$, e per essa Γ_7 è una curva $(4.2-1=)\ 7^{\text{pla}}$ e le R sono rette $(4.1-1=)\ 3^{\text{ple}}$. Inoltre, la Jacobiana sega una qualunque delle Φ in un luogo d'ordine $6.20-2.7.7-1.3.3 = 13$, la cui imagine in ϕ è costituita dai punti fondamentali ab. I sette punti b rappresentano altrettante rette trisecanti di Γ_7; e i punti a rappresentano coniche seganti Γ_7 in cinque punti e appoggiate a due rette R. Perciò la Jacobiana delle Φ è composta: 1° della superficie J_{11} d'ordine 11, luogo delle trisecanti di Γ_7; per essa, Γ_7 è curva quadrupla e le R sono tre generatrici semplici; 2° di tre superficie di 3° ordine $\Psi_3, \Psi_3', \Psi_3''$ (della rete considerata al principio di questa memoria), passanti per Γ_7 e rispettivamente per $R'R''$, $R''R$, RR'. Queste superficie Ψ_3 sono appunto i luoghi delle coniche appoggiate in 5 punti a Γ_7 e incontranti due rette R: come si può facilmente verificare *a posteriori* ricorrendo ad una rappresentazione piana di una Ψ, p. e. ad una projezione dal punto doppio O.*

6. Le superficie costituenti la Jacobiana delle Φ corrispondono alle curve fondamentali dello spazio s: perciò, in questo spazio, avremo una curva c_7 i cui punti corrisponderanno alle generatrici di J_{11}, e il cui ordine sarà 7, perchè ciascuna Φ contiene *sette* generatrici di J_{11}; ed avremo inoltre tre rette r, r', r'', i cui punti corrisponderanno alle

* La rappresentazione ha sei punti fondamentali 1, 2, 3, 4, 5, 6 situati in una conica che è l'imagine del punto doppio. Le imagini delle due rette R e della curva Γ_7 sono p. e. le rette 16, 26 e una curva generale di 4° ordine 12345. La superficie è allora il luogo delle coniche aventi per imagini le coniche 3456.

coniche di Ψ_3, Ψ_3', Ψ_3''. Ciascuna r è una retta (linea d'ordine 1) perchè la corrispondente Ψ_3 ha *una* sola conica in una Φ qualunque. E il sistema omaloidale in s (ossia il sistema delle superficie corrispondenti ai piani di S) sarà per conseguenza costituito dalle superficie f_5 di 5° ordine, aventi in comune c_7 come curva *semplice* (giacchè i suoi punti corrispondono a rette) e le r come rette *doppie* (giacchè i loro punti corrispondono a coniche). Due f_5 si segheranno ulteriormente in una curva k_6 (razionale) d'ordine $5.5-7-3.4=6$; tutte le curve analoghe corrispondono alle rette di S. Una k_6 incontra la c_7 in undici punti e ciascuna r in tre punti, perchè le Jacobiane parziali, corrispondenti a c_7, r, sono rispettivamente dell' ordine 11, 3.

7. La superficie Ψ_3 che contiene Γ_7 ed $R'R''$, insieme con una superficie Ψ, passante per Γ_7 ed R (vi è un fascio di tali superficie Ψ) forma una superficie Φ, cui corrisponderà un piano passante per r; ossia i piani per r corrispondono alle Ψ per Γ_7 ed R; analogamente i piani per r' alle Ψ per Γ_7 ed R'; ed i piani per r'' alle Ψ per Γ_7 ed R''. Questi tre fasci di Ψ, presi due a due, hanno una superficie comune (la Ψ_3 che passa per Γ_7 e per $R'R''$ o per $R''R$ o per RR'); dunque anche i fasci di piani per r, r', r'', presi due a due, hanno un piano comune: ossia le tre rette r, r', r'' giacciono due a due in un medesimo piano. E siccome tutte e tre non possono trovarsi in uno stesso piano (essende rette doppie per superficie di 5° ordine), così esse concorrono in un punto o.

8. Un cono di 2° ordine per $rr'r''$ corrisponderà ad una superficie di 12° ordine $\equiv \Gamma_7^4 R^2 R'^2 R''^2$, dalla quale devono staccarsi le tre superficie $\Psi_3 \equiv \Gamma_7 R'R''$, $\Psi_3' \equiv \Gamma_7 R''R$, $\Psi_3'' = \Gamma_7 RR'$; rimane dunque una superficie di 3° ordine per Γ_7. Alla rete delle superficie Ψ per Γ_7 corrisponde adunque la rete dei coni quadrici per $rr'r''$. Ne segue che alle rette per o corrispondono le coniche per O, appoggiate a Γ_7 in quattro punti. Una di queste coniche incontra un piano arbitrario in due punti; dunque o è un punto triplo per le superficie f_5.

9. Ciascuna delle tre rette r incontra c_7 in tre punti. Infatti la J_{11} e una delle Ψ_3, avendo in comune Γ_7 e due R, si segheranno inoltre in un luogo di 3° ordine: se M è un punto di esso, per M passerà una generatrice di J_{11}, ossia una trisecante di Γ_7. Questa trisecante, avendo così 4 punti comuni con Ψ_3, che è di 3° ordine, giace per intero in questa superficie; dunque quel luogo di 3° ordine, ulteriore intersezione di J_{11} e Ψ_3, è composto di tre rette, trisecanti di Γ_7.* Da ciò segue che c_7 ha tre punti comuni con ciascuna r. Ai cinque punti in cui c_7 incontra ulteriormente un cono quadrico passante per le r corrispondono le cinque rette in cui J_{11} sega una Ψ.

* Ciò può anche dedursi dal fatto che J_{11} è segata da ogni Φ secondo 7 rette, e che p. e. la $\Psi \equiv \Gamma_7 R'R''$ con una $\Psi \equiv \Gamma_7 R$ costituiscono una Φ.

10. Considerando la rappresentazione di una f_5 sul piano corrispondente F in S, le imagini delle sezioni piane di f_5 saranno le sezioni fatte in F dalle Φ, ossia curve di 6° ordine con sette punti doppi $A_1 A_2 \ldots A_7$ (tracce di Γ_7) e tre punti semplici B (tracce delle rette R).* L'imagine della curva fondamentale c_7 sarà una curva $C_{11} \equiv A^4B$, traccia della superficie J_{11}; e le rette r saranno rappresentate da tre cubiche $AB'B''$, $AB''B$, ABB' (tracce delle superficie Ψ_3), passanti pei sette punti A e per due punti B. Ciascuna di queste cubiche è punteggiata in coppie di punti conjugati (mediante le coniche il cui luogo è la relativa Ψ_3), in modo che le curve sestiche A^2B la incontrano soltanto in coppie di punti conjugati.

Le tre cubiche medesime si segano ulteriormente, due a due, in tre punti C, C', C'', uno per ciascun pajo. Ne segue che i punti C, C', C'' costituiscono insieme l'imagine del punto o nel piano F; così che essi giaceranno in ∞^2 sestiche del sistema A^2B. È pure manifesto che $C'C''$, $C''C$, CC' sono coppie di punti conjugati, rispondente per le cubiche $AB'B''$, $AB''B$, ABB'.

11. I punti C sono le intersezioni di F con tre rette, passanti per O e appoggiate a Γ_7, le quali costituiscono l'imagine di o nello spazio S. Tali rette sono quelle nelle quali s'intersecano, due a due, le tre superficie cubiche $\Gamma_7 R'R''$, $\Gamma_7 R''R$, $\Gamma_7 RR'$. Esse corrispondono alle rette R, R', R'' nella corrispondenza univoca che esiste fra le generatrici di J_{11} e le generatrici del cono che da O projetta Γ_7. Ecco come si riconosce tale corrispondenza. Due superficie di 3° ordine passanti per Γ_7 si segano inoltre secondo una conica che passa per O e incontra Γ_7 quattro volte. Ma, se le due superficie passano insieme per una trisecante di Γ_7, esse individuano un fascio, tutte le superficie del quale hanno inoltre in comune una generatrice del cono $O\Gamma_7$: e viceversa, se le due superficie passano per questa generatrice, esse contengono eziandio quella trisecante.

12. I punti fondamenti A, B della rappresentazione di f_5 in F, sono le imagini di sette coniche e di tre rette: queste costituiscono adunque la parte variabile dell' intersezione di una f_5 colla Jacobiana delle f, la quale dev' essere dell' ordine $4(5-1) = 16$ e contenere tre volte la c_7 e sette volte ciascuna delle r. La Jacobiana è dunque composta: 1° della superficie j_{13} d'ordine 13 (perchè le curve C_5 in S incontrano Γ_7 tredici volte), luogo delle coniche appoggiate alle r e seganti c_7 in quattro punti; 2° di tre piani i, i', i'', determinati dalle r prese due a due. Siccome in S la Γ_7 è incontrata rispondente in 3, 5 punti dalle rette e dalle coniche corrispondenti ai punti di c_7, r, così in s per la superficie i_{13} la c_7 è tripla e ciascuna r è quintupla.

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* Cfr. Caporali, l. c.

Il piano $r'r''$ fa parte della Jacobiana delle f in quanto è il luogo delle rette (formanti un fascio) che segano r', r'' e incontrano c_7 nel punto che questa curva ha nel piano all infuori delle r', r''. Il detto piano sega i_{13} secondo tre rette, uscenti dal punto ora nominato di c_7, le quali corrispondono ai punti in cui R è appoggiata a Γ_7.

13. Ai piani per una retta R pello spazio S corrispondono in s superficie di 4° ordine per le quali r è una retta doppia e c_7, r', r'' sono linee semplici. Perciò la curva c_7 giace in tre fasci di superficie di 4° ordine dotate di una retta doppia (una delle r). Un fascio ha per base la retta doppia, la c_7, le altre due rette r e altre tre rette corrispondenti ai punti in cui R è appoggiata a Γ_7.

14. Abbiamo veduto che ad una retta qualunque K in S corrisponde in s una sestica (razionale) k_6 che sega ciascuna r tre volte e la c_7 undici volte. Ma, poichè ai punti di Γ_7 e delle R corrispondono coniche e rette che hanno un determinato numero d'incontri colle linee fondamentali di s, così se K incontra Γ_7 o le R, la corrispondente curva in s si abbassa ad un ordine inferiore. Per es. alle corde di Γ_7 corrispondono le coniche appoggiate in un punto a ciascuna r e in tre punti alla c_7; alle rette seganti le tre R corrispondono le cubiche gobbe che incontrano ciascuna r una volta e la c_7 otto volte; alle rette che incontrano Γ_7 e le tre R corrispondono le rette che segano quattro volte c_7; etc.

Da quest' ultima osservazione si conclude che c_7 ha cinque rette che la segano in quattro punti; giacchè l'iperboloide formato dalle trasversali delle R incontra Γ_7 in $2 \cdot 7 - 3 \cdot 3 = 5$ punti situati fuori delle R medesime.

15. Analoghi risultati si ottengono considerando le rette c nello spazio s e le corrispondenti curve C_5 in S. Alle rette trisecanti di c_7 corrispondono le coniche appoggiate in un punto a ciascuna R e in quattro punti a Γ_7; alle corde di c_7 segate da una r corrispondono le corde di Γ_7 segate da una R; etc.

16. Se ora si suppone data, in uno degli spazi S o s, una superficie passante una o più volte per alcuna delle linee fondamentali, sarà ovvio di ottenere le proprietà della corrispondente superficie nell' altro spazio.

Note on the Pellian Equation. *By* Samuel Roberts, F.R.S.

[*Read March 13th*, 1884.]

1. It was naturally suggested by Euler and Wallis that the solution of the Pellian equation $q^2 - Dr^2 = 1$, where D is a non-quadrate number, might be often made less prolix by using both superior and inferior integer limits as the partial quotients belonging to the continued fraction representing $\sqrt{D}$. On the other hand, Lagrange (*Additions to Euler's Algebra*, Leonard's translation, Art. viii.) objects that, by indiscriminate use of both limits, we run the risk of never arriving at a form whose first coefficient is unity, as we require. This failure will happen, as he points out, whenever we use in the first transformation the inferior limit and then change to the superior limit throughout the succeeding steps of the operation. In fact, starting from a form whose leading coefficient is negative and which has a positive root, and proceeding by substitutions of the kind $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$, where λ is the integer next $>$ the major positive root if both roots are positive, $>$ the positive root if the other is negative, all the leading coefficients of the resulting forms are negative. The form, therefore, which we set out with need not be of definite rank. Positive unity as a first coefficient is thus excluded, but negative unity may, and, in the cases we shall deal with, will occur, when it can be represented by the original form. Of course, there are many other cases in which a similar failure occurs. If, for example, we take for D a prime number, and proceed arbitrarily as to choice of integer limits, until, perchance, we arrive at a form which, by the use of one limit leads to a solution, and if then, instead of employing the successful limit, we pass to the alternative one, no form whose leading coefficient is unity will be arrived at.

Take as a type of the forms met with in such a process (a, b, c). The corresponding quadratic $aX^2 + 2bX + c = 0$ will have a major positive root > 1. Let λ be the integer next greater than this root. If we transform successively the above form by the substitutions $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$ and $\begin{vmatrix} \lambda - 1, & 1 \\ 1, & 0 \end{vmatrix}$, and equate the leading coefficients of the resulting forms, we get

$$a - 2a\lambda - 2b = 0, \quad \text{or} \quad \lambda = -\frac{b}{a} + \frac{1}{2}.$$

Also, k being a positive quantity < 1,

$$\lambda = \frac{-b \pm \sqrt{D}}{a} + k.$$

Therefore a is a factor of $2b$ and $2D$, and $\pm\frac{\sqrt{D}}{a}$ is $<\frac{1}{2}$. Consequently, if D is a prime, $\pm a = 2D$.

For the derivative (a', b', a), by the substitution $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$, we have

$$b' = -a\lambda - b = -\frac{a}{2} = \mp D,$$

$$a' = \frac{b'^2 - D}{a} = \pm\frac{D-1}{2},$$

numerically >1 unless $D = 3$. But the conclusion holds also for 3, since 6 does not occur as a leading coefficient.

2. I shall now show that, if we always take the nearest integer limit, a form will be arrived at whose first coefficient is unity.*

The Pellian equation will, therefore, be solved, and the number of steps is in this case usually reduced.

We set out with the form $q^2 - Dr^2$. If the superior limit is the nearer one to $\sqrt{D}$, our first transformation is

$$(\lambda^2 - D, \ -\lambda, \ 1).$$

The substitution is $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$, and ω, ω', the roots of the corresponding quadratic in order of magnitude, are both positive.

Since the substitution gives

$$\lambda - \sqrt{D} = \frac{1}{\omega}, \quad \lambda + \sqrt{D} = \frac{1}{\omega'},$$

ω is >2, and $\omega' < \frac{1}{2}$.

If the superior limit is the nearer one, the first transformation is

$$[-(D-\lambda^2), \ \lambda, \ 1].$$

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* I am sorry that I overlooked, until the time for sending this paper to the press was at hand, two memoirs, which anticipate me in showing that the Pellian equation is solved by using nearest limits exclusively or superior limits exclusively. The papers I refer to are—

I. "Ueber die Eigenschaften der periodischen negativen Kettenbruche, welche die Quadratwurzel aus einen ganzen positiven Zahl darstellen," von M. A. Stern. (*Abhandlungen der Kön. Gesells. d. Wiss. zu Göttingen*, Band 12, 1866, pp. 3—48.)

II. "Ueber einer neuen Methode die Pellische Gleichung aufzulösen," von B. Minnigerode. (*Nachr. der Kön. Gesells. d. Wiss. zu Göttingen*, 1873, pp. 619—642.)

It became a question whether the present paper should be withdrawn or retained. As the methods employed are somewhat different and some further results are given, it has been thought that the publication of the paper may still be acceptable to English readers. In an addition I give a brief account of the German memoirs, previous knowledge of which would, of course, have materially modified the form and arrangement of my paper.

The substitution is $\begin{vmatrix} \lambda, & 1 \\ 1, & 0 \end{vmatrix}$, and, if ω, ω' are the roots in order of magnitude of the corresponding quadratic, we have

$$\sqrt{D}-\lambda = \frac{1}{\omega} < \frac{1}{2}, \quad -\sqrt{D}-\lambda = \frac{1}{\omega'},$$

so that ω is >2, and ω' is negative and $<\frac{1}{2}$ numerically.

If, in like manner, any form which we reach, still following the major positive root, is (a, b, c), whose corresponding root ω is >2 and positive, either substitution $\begin{vmatrix} \lambda, & -1 \\ 1, & 0 \end{vmatrix}$ or $\begin{vmatrix} \mu, & 1 \\ 1, & 0 \end{vmatrix}$ gives a left associate form (a', b', a) which will have a corresponding root >2 and positive. For, if ω, ω' are the roots corresponding to the form (a, b, c), and Ω, Ω' those corresponding to the derivative form, we have either

$$\lambda-\omega = \frac{1}{\Omega} < \frac{1}{2} \text{ or } \omega-\mu = \frac{1}{\Omega} < \frac{1}{2}.$$

It appears then that, except in the first transformations for $D=2$ and $D=3$, viz., $(q+r)^2-2q^2$ and $(2q-r)^2-3q^2$, we shall never have occasion to make the substitutions $\begin{vmatrix} 2, & -1 \\ 1, & 0 \end{vmatrix}$ or $\begin{vmatrix} 1, & 1 \\ 1, & 0 \end{vmatrix}$, or, of course, $\begin{vmatrix} 1, & -1 \\ 1, & 0 \end{vmatrix}$.

That is to say, for inferior limits λ is >1, and for superior limits λ is >2.

Consequently, in the resulting continued fraction representing $\sqrt{D}$, no denominator will be <2 and no denominator immediately followed by a negative unit numerator will be <3. It will be understood that I attach signs to the numerators only.

3. We can now have recourse to the process for converting a continued fraction with mixed negative and positive numerators into one with positive numerators exclusively.

This depends on the identity

$$-\frac{b}{a+R} = -1+\frac{1}{1}+\frac{b}{a-b+R},$$

where R may denote the residue of the continued fraction after the term $-\frac{b}{a}$. Therefore

$$a-\frac{1}{a_1}-\frac{1}{a_2+R} = a-1+\frac{1}{1}+\frac{1}{a_1-2}+\frac{1}{1}+\frac{1}{a_2-1+R},$$

$$a-\frac{1}{a_1}+\frac{1}{a_2}-\frac{1}{a_3+R} = a-1+\frac{1}{1}+\frac{1}{a_1-1}+\frac{1}{a_2-1}+\frac{1}{1}+\frac{1}{a_3-1+R},$$

and so forth.

In this way we can provide for the case in which some numerators are positive and some negative. No partial quotient will become zero in the transformation unless an original partial quotient is unity or 2, immediately followed in the last case by a negative numerator.

4. Now, in the proposed process, we have seen that no partial quotient is unity or 2, immediately followed in the last case by a negative numerator. Moreover, it is accepted that a continued fraction can only be developed in one form with positive unit numerators and denominators >0 and integer, *i.e.* in the normal form.*

This being so, the continued fraction, which we obtain by taking the nearer integer limits, is converted into the normal form (1) by the introduction of fractional elements $1/1$, (2) by diminishing certain denominators by 1 or 2.

Hence we shall always have a partial quotient 2λ or $2\lambda+1$ or $2\lambda+2$, where λ is the greatest integer in $\sqrt{D}$; because we know that the partial quotient 2λ occurs in the normal development.

The denominator in such case is derived from a form having positive or negative unity for its first coefficient. For limits involving this can be established to the values of the middle coefficients of the forms. Thus, if a form has a negative root, the middle coefficient is numerically $< \sqrt{D}$, and $\omega < \frac{2\sqrt{D}}{a}$ numerically. If both roots are positive, one will be >2 and the other $<\frac{1}{2}$, and we may, without detriment to the argument, suppose the first coefficient to be positive and the form $(a, -b, c)$. To produce a denominator $=$ or $>2\lambda$, we must at least have $\omega > 2\lambda - \frac{1}{2}$, and because $\omega - \omega' = \frac{2\sqrt{D}}{a}$, $2\lambda-1$ is $< \frac{2\sqrt{D}}{a}$. If a be greater than unity, we must have $\lambda < 2$. The cases in which $\lambda = 1$, viz., $D = 2$, $D = 3$, may be separately verified.†

5. There may be, and in reality are, numbers for which the nearer limits are always inferior. There are also numbers for which the nearer limits are always superior. The former class is signalised by the absence of unit denominators in the normal development.

The economy of terms is not however measured directly by counting the unit partial quotients in the normal form. If there are $2n$ such units in consecutive order, we only remove n terms; if $2n+1$ such units, we remove $n+1$ terms.

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* There is a generally trivial exception in the case of finite continued fractions. If the last fractional element is $1/m$, we may write for this $1/m-1+1/1$. This modification takes effect in the following theorem. If P, Q, R are positive integers in order of magnitude, and $Q^2-PR = \pm 1$, P/Q can be expressed by a normal symmetric continued fraction; *e.g.*, $7^2-8.6 = 1$ and $8/7$ is expressed by $1+1/7$, but also by $1+1/6+1/1$, which justifies the theorem in this case.

† It follows that the development under consideration affords the same series of solutions as the normal development and in the same order.

Unit denominators may have been derived from 2—1, or 3—2 occurring in the transformation to the normal form.

The prime 1153 affords a fair instance of abbreviation. In the usual period there are 43 terms and 26 units. Taking the line of nearest approximation, there are 24 terms in the period. The first period, of course, gives the least solution of $q^2-1153r^2=-1$. The prime 97 is also a good instance.

In this process what we really effect is the removal of the elements 1/1 from the normal form. This we could do directly, if the normal form were given, by the identity

$$\frac{1}{1+\dfrac{1}{a+R}}=1-\frac{1}{a+1+R},$$

and it is a method given by Lagrange himself for abridging continued fractions in certain cases.*

6. I shall further show that, if we proceed by using superior limits only, the notation of the Pellian equation will be duly obtained by means of a leading coefficient equal to unity.

Our first transformation will now be

$$(\lambda^2-D,\ -\lambda,\ 1),$$

and, if ω, ω' are the corresponding quadratic roots in order of magnitude, ω is $>$ and $\omega'<1$, both being positive. The next transformation gives a quadratic whose roots are determined by

$$\mu-\omega=\frac{1}{\Omega},\quad \mu-\omega'=\frac{1}{\Omega'},$$

$\Omega>1$ and $\Omega'<1$, both being positive, since μ cannot be <2.

In like manner, every succeeding form will have two positive roots, one >1 and the other <1. Each form has only one left associate, and only one right associate by the law of formation. For, if we derive a form by $\quad \mu-\pi=\dfrac{1}{\Pi},\quad \mu-\pi'=\dfrac{1}{\Pi'}$,

π, π' being the roots in order of magnitude corresponding to the given form, and Π, Π' in order of magnitude being those of the derived form, we go back from the derived form without ambiguity by

$$\mu-\frac{1}{\Pi}=\pi,\quad \mu-\frac{1}{\Pi'}=\pi',$$

and the substitution $\begin{vmatrix} 0, & 1 \\ 1, & \mu \end{vmatrix}$.

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* *Traité de la Résolution des équations numériques*, Cap. vi., Art. 111, where it is applied to the ratio of the circumference of a circle to its diameter; or else *Additions to Euler's Algebra*, Chap. I.

If, then, we can show that the forms are finite in number, a period will be established. The right associate of the form $(\lambda^2-D, -\lambda, 1)$ is $(1, -\lambda, \lambda^2-D)$, so that the solution of the Pellian equation will follow.

Now, when μ is >2, the corresponding forms exist in finite number only, because then we have

$$\Omega-\Omega' = \frac{2\sqrt{D}}{a} > 1, \text{ or } a \text{ is } < 2\sqrt{D} \text{ and } b \text{ is } < a+\sqrt{D} < 3\sqrt{D}.$$

If however $\mu = 2$, we are unable to fix limits in this way, since $\frac{2\sqrt{D}}{a}$ may be <1.

But the substitutions $\begin{vmatrix} 2, & -1 \\ 1, & 0 \end{vmatrix}$ cannot follow in infinite succession,

since $$1-\frac{1}{2-}\frac{1}{2-}\frac{1}{2}- \&c. \textit{ ad inf.}$$

is a development representing zero; it follows that there must be an infinite number of interruptions of the series by partial quotients exceeding 2. Since there is a limited number of forms giving such partial quotients, the series is periodic, and, each form having only one left and one right associate by the law of formation, the period includes the form $q^2-2\lambda q^2+(\lambda^2-D)\,r^2$.

The use of superior limits alone, however, may prolong the operation, and moreover does not directly furnish the solution of $q^2-Dr^2=-1$ when possible, since every leading coefficient is positive.

In one passage of his Algebra, Wallis expresses himself unguardedly, and lays himself open to the criticism of Lagrange. Yet his more formal statement appears to be correct. Referring to expedients for abridging the process, Dr. Wallis says:—"First, whereas in the former process, when a quantity fell between two limits, we always made use of the *lesser* limit (making addition to it), we might, if we had pleased, have always made use of the *greater* (making defalcation from it). But the most expedient way for shortening the process is to make use sometime of the one and sometime of the other, according as this or that comes nearest the truth."

It is nevertheless pretty clear that Wallis possessed no satisfactory proof to support his statement. I am inclined to think also that the advantage of any abridgment gained by these methods is slighter than was supposed, since we lose symmetry, or at least the symmetry is less familiar.

7. In the concrete example furnished by Lagrange, where q^2-6r^2 is the form, and the first transformation is by inferior limit, the rest

by superior limit, and in other instances of the same kind, the transformation to positive unit numerators introduces zero denominators. If we eliminate these by summation, we get the normal development.

In these cases, the term of the normal form $1/2\mu$, where μ is the greatest integer in $\sqrt{D}$, is disguised by decomposition. Its place is, however, indicated by a series of denominators 2 followed immediately by negative numerators. Thus $\sqrt{D}$ is represented by

$$2+\frac{1}{3}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{4}-\frac{1}{2}-\&c.$$

Diminishing 3 by unity, $2+\frac{1}{2}$ gives a solution of $q^2-6r^2=1$. In like manner, though inconveniently, we could obtain a solution in other such cases. The form which commences the series representing $1/2\mu$ in a decomposed shape is characterised by the algebraical sum of the coefficients being equal to unity, and $q=1$, $r=1$ gives a representation of unity accordingly; so that a solution is furnished by this indication. Terms $-1/1$ cannot exist in successive series, being always immediately followed by a positive numerator, since the substitution $\begin{vmatrix} 1, & -1 \\ 1, & 0 \end{vmatrix}$ does not occur.

A process by integer limits, which does not furnish a series of terms $-1/2$ capable of accounting for $1/2\mu$, must include that term or $1/(2\mu+1)$ or $1/(2\mu+2)$. A variety of developments leading to a solution of the Pellian equation may therefore be determined. We can proceed, for instance, as far as we like by inferior limits, and then change to nearest limits, or, commencing arbitrarily as to choice of limits, we may change to inferior or nearest limits; or we may proceed alternately with inferior and superior limits; or, after reaching a form with a positive leading coefficient, we may proceed with superior limits. If, in fact, we go astray up to any point, we can regain the correct path, though possibly we may miss the least solution.

On the whole, then, it seems that the process by integer limits is theoretically arbitrary as to the choice of those limits. For in those cases in which a solution of the Pellian equation is not afforded by a form whose leading coefficient is unity, forms occur in which the algebraical sum of the coefficients is positive unity. And the conclusion is similar with respect to negative unity, when it can be represented.

8. If a is $< \sqrt{D}$, and the equation $q^2-Dr^2=\pm a$ is resolvable, q, r being relatively prime, $\pm a$ will be found as a leading coefficient in the forms belonging to the normal development of $\sqrt{D}$, or, as otherwise stated, a is the denominator of a complete quotient $\frac{\sqrt{D}+b}{a}$. We have

seen that, in the development of $\sqrt{D}$ by superior limits, the leading coefficients will be all positive. If, however, $q^2-Dr^2=+a$ is resolvable, and a is $<2\sqrt{D}$, a will be a leading coefficient of a derived form.

We can develop a fraction $\frac{p}{q}$, where p and q are integers, in the form

$$a-\frac{1}{b}-\frac{1}{c}-\frac{1}{d}-\ldots-\frac{1}{k},$$

a form which will be finite if we take the complete quotient when the denominator is unity. If we still continued to take the superior limit, the remainder of the continued fraction would have to be written *ad infinitum* in the form $-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\ldots$ which is $=-1$

Taking then the finite form, suppose that $\frac{p_0}{q_0}$ is the reduced fraction next preceding $\frac{p}{q}$.

If now $\frac{p}{q}-\sqrt{D}=\frac{1}{q(q\mu-q_0)}$, where μ is >1, we have, since $pq_0-p_0q=-1$ in this case,

$$\mu=\frac{1+pq_0-\sqrt{D}qq_0}{q(p-q\sqrt{D})}=\frac{p_0-\sqrt{D}q_0}{p-\sqrt{D}q}$$
$$=\frac{pp_0-qq_0D+\sqrt{D}}{p^2-q^2D};$$

but this is the form of the complete quotient corresponding to $\frac{p}{q}$, $\frac{p_0}{q_0}$, on the supposition that the development in question is that of $\sqrt{D}$ by superior limits. Hence the only condition to be fulfilled is

$$p-q\sqrt{D}<\frac{1}{q-q_0},$$

q_0 being always $<q$.

We have also, suppose,

$$p-q\sqrt{D}=\frac{a}{p+q\sqrt{D}},$$

and the condition is

$$p+q\sqrt{D}-aq+aq_0>0.$$

But p is $>q\sqrt{D}$ and $2q\sqrt{D}-aq+aq_0>0$ if a is $<2\sqrt{D}$.

To provide similarly for the case of $q^2-Dr^2=-a$, where a is $<2\sqrt{D}$. We commence the development of $\sqrt{D}$ by taking the inferior limit and then proceed with superior limits. The leading coefficients of the forms will then be all negative. Following the same reasoning as before, we get for the condition that $\frac{p}{q}$, and $\frac{p_0}{q_0}$, the next preceding reduced fraction of the corresponding development of $\frac{p}{q}$, may belong

to the development of $\sqrt{D}$,

$$q\sqrt{D}-p<\frac{1}{q-q_0},$$

and also we have $$q\sqrt{D}-p=\frac{a}{p+q\sqrt{D}}.$$

The condition to be fulfilled is again

$$p+q\sqrt{D}-aq+aq_0>0,$$

or $$2q\sqrt{D}-\frac{a}{p+q\sqrt{D}}-aq+aq_0>0,$$

or $$2q\sqrt{D}-a\left(q-q_0+\frac{1}{p+q\sqrt{D}}\right)>0,$$

which is so if a is $<2\sqrt{D}$ and $q_0>0$.

By the two developments, then, we solve, when possible, $q^2-Dr^2=\pm a$, $a<2\sqrt{D}$. The limit to which the normal development is subject is therefore doubled.

9. We will consider more particularly the case in which the first transformation is by inferior limit and the succeeding ones are by superior limits.

Then q^2-Dr^2 becomes in the first instance

$$-(D-\lambda^2)q^2+2\lambda qr+r^2,$$

where λ is the greatest integer in $\sqrt{D}$.

If κ be the integer next $>\frac{1}{\sqrt{D}-\lambda}$, we get, by the substitution

$$\begin{vmatrix}\kappa, & -1\\ 1, & 0\end{vmatrix};$$

$$-(D\kappa^2-\overline{\kappa\lambda+1}^2)q^2+2[(D-\lambda^2)\kappa-\lambda]qr-(D-\lambda^2)r^2\ldots\ldots(a).$$

Now put m for the value of this when $q=1$, $r=1$, so that

$$-(D-\lambda^2)(\kappa-1)^2+2\kappa\lambda+1-2\lambda=m,$$

m being positive. Then

$$-(D\kappa^2-\overline{\kappa\lambda+1}^2)m=[(D-\lambda^2)\kappa(\kappa-1)-2\kappa\lambda-1+\lambda]^2-D=\Pi^2-D,$$

$$-(D-\lambda^2)m=[(D-\lambda^2)(\kappa-1)-\lambda]^2-D=(\Pi+m)^2-D,$$

$$[(D-\lambda^2)\kappa-\lambda]m=D-\Pi(\Pi+m),$$

or the derived form is

$$-\frac{D-\Pi^2}{m}q^2+2\frac{D-\Pi(\Pi+m)}{m}qr-\frac{D-(\Pi+m)^2}{m}r^2.$$

If $\frac{m}{\sqrt{D}+\Pi}$ is <1, we transform again by the substitution $\begin{vmatrix}2, & -1\\ 1, & 0\end{vmatrix}$,

obtaining $-\frac{D-(\Pi-m)^2}{m}q^2+2\frac{D-(\Pi-m)\Pi}{m}qr-\frac{D-\Pi^2}{m}r^2$,

and we may proceed in this way, as long as the leading coefficient remains negative, so that $\rho m-\Pi$ is $<\sqrt{D}$.

When the next leading coefficient would be positive under this process, let us write $\frac{D-\nu^2}{m}$ for $\frac{D-(\Pi-\rho m)^2}{m}$, and the form is, if

$$\nu=\rho m-\Pi,$$

$$-\frac{D-\nu^2}{m}q^2+2\frac{D-\nu(\nu-m)}{m}qr-\frac{D-(\nu-m)^2}{m}r^2;$$

and, if k is the integer next $>\frac{m}{\sqrt{D}-\nu}$, the derived form will be

$$-\left(\frac{Dk^2-(k\nu+m)^2}{m}\right)q^2+\frac{2[(D-\nu^2)k-m\nu]}{m}qr-\frac{D-\nu^2}{m}r^2 \ldots\ldots (a'),$$

which, as in the preceding case, becomes

$$-\frac{D-\Pi_1^2}{m_1}q^2+\frac{2[D-\Pi_1(\Pi_1+m_1)]}{m_1}qr-\frac{D-(\Pi_1+m)^2}{m_1}r^2,$$

m_1 being positive; and we see that (a') is the same in form as (a), when $m=1$.

The structure of the series of forms is now clear, and, since

$$Q^2-DR^2=-D$$

is always possible, in integers, there will be a leading coefficient, $-D$, in the period, and all the others will be negative and $<D$ numerically, the last leading coefficient being $-(D-\lambda^2)$, where λ is the greatest integer in $\sqrt{D}$.

Moreover, the general form of the denominators of which m_1 is the type is

$$-\frac{D-\nu^2}{m}(k-1)^2-2\nu+2\nu k+m.$$

If we substitute for $k-1$ its value $\frac{m}{\sqrt{D}-\nu}-\sigma$, where σ is positive and <1, we get

$$2\sigma\sqrt{D}-\sigma^2\frac{D-\nu^2}{m}=m_1<2\sqrt{D}.$$

If for $\frac{D-\nu^2}{m}$ we write a, ν being as above a maximum [*i.e.*, $D-(\nu+m)^2$ is negative], then $D^2=\nu^2+am$, where not only are 2ν, a and $m<2\sqrt{D}$ respectively, but $\nu+m$ is by supposition $>\sqrt{D}$ and also $\nu+a>\sqrt{D}$.

For, $a=\frac{D-\nu^2}{m}=(\sqrt{D}-\nu)\left(\frac{\sqrt{D}+\nu}{m}\right)$ and $\frac{\sqrt{D}+}{m}$ is >1, since the corresponding minor root is $1-\frac{m}{\sqrt{D}+\nu}$ and is positive and <1.

These limiting coefficients therefore furnish values of a, which belong to a system $-a, \nu, m$ which are the numerical values of the coefficients of a reduced form. Such values of a would, therefore, be furnished by the normal development. But there may be cases in which $\frac{D-\nu^2}{m}$ is $< 2\sqrt{D}$, but ν is not a maximum relative to m, and these cannot be furnished by the normal development. The forms which arise when we use superior limits only, may be similarly investigated. The case is of course closely analogous to the one we have been discussing. The development when nearer limits are employed is also perhaps worth more detailed treatment, and it is not unlikely that analogous deviation from the normal procedure might be useful in the general theory of binary quadratic forms.

I append some numerical examples, reference to which may facilitate the reading of the previous articles.

By superior limits, we have, for q^2-13r^2,

$$\begin{aligned}
(4q-r)^2-13q^2 &= 3q^2-8qr+r^2,\\
3(3q-r)^2-8q(3q-r)+q^2 &= 4q^2-10qr+3r^2,\\
4(3q-r)^2-10q(3q-r)+3q^2 &= 9q^2-14qr+4r^2,\\
9(2q-r)^2-14q(2q-r)+4q^2 &= 12q^2-22qr+9r^2,\\
12(2q-r)^2-22q(2q-r)+9q^2 &= 13q^2-26qr+12r^2,\\
13(2q-r)^2-26q(2q-r)+12q^2 &= 12q^2-26qr+13r^2,\\
12(2q-r)^2-26q(2q-r)+13q^2 &= 9q^2-22qr+12r^2,\\
9(2q-r)^2-22q(2q-r)+12q^2 &= 4q^2-14qr+9r^2,\\
4(3q-r)^2-14q(3q-r)+9q^2 &= 3q^2-10qr+4r^2,\\
3(3q-r)^2-10q(3q-r)+4q^2 &= q^2-8qr+3r^2,\\
\&c. &\quad \&c.
\end{aligned}$$

All the leading coefficients are positive,* but, observing that the sum of the coefficients of $9q^2-14qr+4r^2$ is negative unity, we have

$$4-\frac{1}{3}-\frac{1}{2}=18/5,$$

giving the least solution of $q^2-13r^2=-1$.

The theorem of Professor Smith (*Proceedings*, Vol. vii., p. 197) will apply to these processes as well as to the normal form. Thus, proceeding in the case of q^2-13r^2 with the nearest limits, the product

* In general, for superior limits, $\frac{Q}{R}-\sqrt{N}$ is positive, if $\frac{Q}{R}$ is a reduced fraction.

of the three complete quotients $\frac{4+\sqrt{13}}{3}$, $\frac{5+\sqrt{13}}{4}$, $\frac{3+\sqrt{13}}{1}$ is equal to $18+5\sqrt{13}$, giving the above solution of $q^2-13r^2=-1$.

If we take q^2-44r^2, the superior limits are also the nearer ones, and we have the forms

$$q^2-44r^2,$$
$$(5, -7, 1)*,$$
$$(4, -8, 5),$$
$$(5, -8, 4),$$
$$(1, -7, 5)*.$$

The product $\frac{7+\sqrt{44}}{5}\cdot\frac{8+\sqrt{44}}{4}\cdot\frac{8+\sqrt{44}}{5}\cdot\frac{7+\sqrt{44}}{1}$

gives $199+30\sqrt{44}$ and $q=199$, $r=30$ constitute the lowest solution of $q^2-44r^2=1$.

In the following numerical example, I write down the substitution opposite the form on which it operates, so that each form is derived from the preceding one.

Taking the form q^2-151r^2, we have for the normal development

(	1,	0,	−151)	$12q+r, q$	(−15,	11,	2)	$q+r, q$
*(	−7,	12,	1) ..... *	$3q+r, q$	(	9,	−4,	15)	$q+r, q$
(	10,	−9,	−7)	$2q+r, q$	(−14,	5,	9)	$q+r, q$
(	−3,	11,	10)	$7q+r, q$	(	5,	−9,	−14)	$4q+r, q$
(	17,	−10,	−3)	$q+r, q$	(	−6,	11,	5)	$3q+r, q$
(	−6,	7,	17)	$3q+r, q$	(	17,	−7,	−6)	$q+r, q$
(	5,	−11,	−6)	$4q+r, q$	(	−3,	10,	17)	$7q+r, q$
(−14,	9,	5)	$q+r, q$	(	10,	−11,	−3)	$2q+r, q$
(	9,	−5,	−14)	$q+r, q$	(	−7,	9,	10)	$3q+r, q$
(−15,	4,	9)	$q+r, q$	*(	1,	−12,	−7)	$*24q+r, q$
(	2,	−11,	−15)	$11q+r, q$				

The asterisks indicate the periods.

Employing nearest limits for the same form, we have

(	1,	0,	−151)	$12q+r, q$	(	9,	−13,	2)	$3q-r, q$
*(	−7,	12,	1) ..... *	$3q+r, q$	(	5,	−14,	9)	$5q+r, q$
(	10,	−9,	−7)	$2q+r, q$	(	−6,	11,	5)	$4q-r, q$
(	−3,	11,	10)	$8q-r, q$	(	−3,	13,	−6)	$8q+r, q$
(	−6,	13,	−3)	$4q+r, q$	(	10,	−11,	−3)	$2q+r, q$
(	5,	−11,	−6)	$5q-r, q$	(	−7,	9,	10)	$3q+r, q$
(	9,	−14,	5)	$3q-r, q$	*(	1,	−12,	−7)	$*24q+r, q$
(| 2, | −13, | 9)	$13q-r, q$					

Using superior limits, we get

(1,	0, −151)	$13q-r$,	q
*(18,	−13, 1)	* $2q-r$,	q
(21,	−23, 18)	$2q-r$,	q
(10,	−19, 21)	$4q-r$,	q
(29,	−21, 10)	$2q-r$,	q
(42,	−37, 29)	$2q-r$,	q
(49,	−47, 42)	$2q-r$,	q
(50,	−51, 49)	$2q-r$,	q
(45,	−49, 50)	$2q-r$,	q
(34,	−41, 45)	$2q-r$,	q
(17,	−27, 34)	$3q-r$,	q
(25,	−24, 17)	$2q-r$,	q
(21,	−26, 25)	$2q-r$,	q
(5,	−16, 21)	$6q-r$,	q
(9,	−14, 5)	$3q-r$,	q
(2,	−13, 9)	$13q-r$,	q

(9,	−13, 2)	$3q-r$,	q
(5,	−14, 9)	$6q-r$,	q
(21,	−16, 5)	$2q-r$,	q
(25,	−26, 21)	$2q-r$,	q
(17,	−24, 25)	$3q-r$,	q
(34,	−27, 17)	$2q-r$,	q
(45,	−41, 34)	$2q-r$,	q
(50,	−49, 45)	$2q-r$,	q
(49,	51, 50)	$2q-r$,	q
(42,	47, 49)	$2q-r$,	q
(29,	−37, 42)	$2q-r$,	q
(10,	−21, 29)	$4q-r$,	q
(21,	−19, 10)	$2q-r$,	q
(18,	−23, 21)	$2q-r$,	q
*(1,	−13, 18)	*$26q-r$,	q

From the above we gather that the positive numbers $<2\sqrt{151}$ represented by q^2-151r^2 are

1, 2, 5, 9, 10, 17, 18, 21.

Transforming first of all by inferior limits, and afterwards by superior limits, we get

				For q.	For r.
(	1,	0, −151)		$12q+r$,	q
(	−7,	12, 1)		$4q-r$,	q
*(	−15,	16, −7)	*	$2q-r$,	q
(	−3,	14, −15)		$9q-r$,	q
(	−6,	13, −3)		$5q-r$,	q
(	−23,	17, −6)		$2q-r$,	q
(	−30,	29, −23)		$2q-r$,	q
(	−27,	31, −30)		$2q-r$,	q
(	−14,	23, −27)		$3q-r$,	q
(	−15,	19, −14)		$2q-r$,	q
(	−35,	26, −15)		$2q-r$,	q
(	−51,	44, −35)		$2q-r$,	q
(	−63,	58, −51)		$2q-r$,	q
(	−71,	68, −63)		$2q-r$,	q
(	−75,	74, −71)		$2q-r$,	q
(	−75,	76, −75)		$2q-r$,	q
(	−71,	74, −75)		$2q-r$,	q
(	−63,	68, −71)		$2q-r$,	q
(	−51,	58, −63)		$2q-r$,	q
(	−35,	44, −51)		$2q-r$,	q

				For q.	For r.
(	−15,	26, −35)		$3q-r$,	q
(	−14,	19, −15)		$3q-r$,	q
(	−27,	23, −14)		$2q-r$,	q
(	−30,	31, −27)		$2q-r$,	q
(	−23,	29, −30)		$2q-r$,	q
(	−6,	17, −23)		$5q-r$,	q
(	−3,	13, −6)		$9q-r$,	q
(	−15,	14, −3)		$2q-r$,	q
(	−7,	16, −15)		$5q-r$,	q
(	−30,	19, −7)		$2q-r$,	q
(	−51,	41, −30)		$2q-r$,	q
(	−70,	61, −51)		$2q-r$,	q
(	−87,	79, −70)		$2q-r$,	q
(−102,		95, −87)		$2q-r$,	q
(−115,		109, −102)		$2q-r$,	q
(−126,		121, −115)		$2q-r$,	q
(−135,		131, −126)		$2q-r$,	q
(−142,		139, −135)		$2q-r$,	q
(−147,		145, −142)		$2q-r$,	q
(−150,		149, −147)		$2q-r$,	q

	For q.	For r.		For q.	For r.
$(-151, 151, -150)$	$2q-r$,	q	$(-102, 109, -115)$	$2q-r$,	q
$(-150, 151, -151)$	$2q-r$,	q	$(-87, 95, -102)$	$2q-r$,	q
$(-147, 149, -150)$	$2q-r$,	q	$(-70, 79, -87)$	$2q-r$,	q
$(-142, 145, -147)$	$2q-r$,	q	$(-51, 61, -70)$	$2q-r$,	q
$(-135, 139, -142)$	$2q-r$,	q	$(-30, 41, -51)$	$2q-r$,	q
$(-126, 131, -135)$	$2q-r$,	q	*$(-7, 19, -30)$	* $5q-r$,	q
$(-115, 121, -126)$	$2q-r$,	q			

From which it appears that the numbers $< 2\sqrt{151}$ represented by q^2-151r^2 negatively are

$$3, \quad 6, \quad 7, \quad 14, \quad 15, \quad 23.$$

I have written out the example fully, and in several forms for the purpose of comparison.

As an example involving an arbitrary number, we may take

$$D = \overline{4n+1}^2+\overline{3n+1}^2 = 25n^2+14n+2.$$

The normal development of $\sqrt{D}$ is

$$5n+1+\frac{1}{2}+\frac{1}{2}+\frac{1}{10n+2}+\ldots.$$

Proceeding by superior limits, we have

$[1, \quad 0, \quad -(25n^2+14n+2)]$,	$(5n+2)\,q-r$, $\quad q$,
$[6n+2, \quad -(5n+2), \quad 1]$*,	$*2q-r$, $\quad q$,
$[4n+1, \quad -(7n+2), \quad 6n+2]$,	$4q-r$, $\quad q$,
$[14n+2, \quad -(9n+2), \quad 4n+1]$,	$2q-r$, $\quad q$,
... ...	... ...
$\big[14n+10kn-(k^2-2)$,	
$-14n+5\,(2k-1)\,n-2k^2+2k+4$,	$2q-r$, $\quad q$,
$14n+10\,(k-1)\,n-\overline{k-1}^2+2\big]$,	

until $k = 10n+1$ gives

$[4n+1, \quad -(9n+2), \quad 14n+2]$,	$4q-r$, $\quad q$,
$[6n+2, \quad -(7n+2), \quad 4n+1]$,	$2q-r$, $\quad q$,
$[\quad 1, \quad -(5n+2), \quad 6n+2]$*,	$*(10n+4)\,q-r$, $\quad q$.

Taking the inferior limit at the first step, and afterwards superior limits, we get

$[1, \quad 0, \quad -(25n^2+14n+2)]$,	$(5n+1)\,q+r$, $\quad q$,
$[-(4n+1), \quad 5n+1, \quad 1]$,	$3q-r$, $\quad q$,
$[-(6n+2), \quad 7n+2, \quad -(4n+1)]$,	$2q-r$, $\quad q$,
$[\quad -1, \quad 5n+2, \quad -(6n+2)]$,	$(10n+4)\,q-r$, $\quad q$,
&c.,	&c.

The sequel is indicated by the result for superior limits exclusively.

Addition I.

1. Dr. Stern, in his memoir before mentioned, compares the continued fraction $a+\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_m}$ with one of equal value of the form

$$b-\frac{1}{b_1}-\frac{1}{b_2}-\ldots-\frac{1}{b_n}.$$

The first form is called positive and denoted by $(a, a_1, a_2, \ldots a_m)$, the second is called negative and denoted by $[b, b_1, b_2, \ldots b_n]$.

By means of the identity

$$a+\frac{1}{a_1}+\frac{1}{R}=a+1-\frac{1}{2}-\frac{1}{1}+\frac{1}{a_1-2}+\frac{1}{R},$$

[It seems simpler to take the identity

$$a+\frac{1}{a_1}+\frac{1}{R}=a+1-\frac{1}{1}+\frac{1}{a_1-1}+\frac{1}{R}]$$

the author obtains the following rule:—Let $\overline{a_k-1}$ denote a succession of a_k-1 denominators each $=2$, then we have generally

$$(a, a_1 \ldots a_{2k}) = [a+1, \overline{a_1-1}, a_2+2, \overline{a_3-1} \ldots a_{2k}+1],$$

$$(a, a_1 \ldots a_{2k+1}) = [a+1, \overline{a_1-1}, a_2+2 \ldots \overline{a_{2k+1}-1}].$$

When k is infinite, the forms coalesce.

The periodicity of the positive fraction involves that of the negative fraction, and the author compares in detail the denominators and numerators of the complete quotients belonging to a positive and a negative fraction, representing in each case the square root of the same non-quadrate integer A.

The method therefore is the same in principle as that employed in §§ 3, 4, in reference to the continued fraction obtained by the nearest limits.

The negative period is evidently analogous to that of the continued fraction obtained by first of all using the inferior limit, and afterwards the superior limits—a case briefly treated in § 9. Thus no leading coefficient of a form is $>A$, and no middle coefficient $>2A$; and, if a leading coefficient is $=A$, then $x^2-Ay^2=-1$ is resoluble, as, of course, is obvious.

The author notes that, if $x^2-Ay^2=d_n$, when d_n is a positive integer $<\sqrt{A}$, then d_n will appear as the denominator of a complete quotient

of the negative continued fraction. I do not find it remarked, that the same is true as long as d_n is $< 2\sqrt{A}$.*

The cases in which the period has one mid-term or two mid-terms are discussed, and it is shown that, when there are two mid-terms, A is the difference of two squares determined.

Analogues to Goepel's properties of the normal fraction and other interesting results are given, and the memoir concludes with a table for non-quadrate numbers up to 100, for the negative development of $\sqrt{A}$, similar to Degen's "Canon Pellianus."

2. Dr. Minnigerode notices that Herr Stern had shown how the continued fraction with negative unit numerators and positive integer denominators, representing the square root of a non-quadrate number, may be used to solve the corresponding Pellian equation.

The author then proceeds to prove that the equivalent continued fraction with negative denominators may also be used when the integer absolutely nearest to the irrational is taken as the limit.

He treats the subject from the point of view afforded by the theory of Binary Quadratic Forms.

Each substitution $\begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix}$, where $\alpha\delta - \beta\gamma = 1$, which transforms (a, b, c), determinant D, of the species σ, into itself, yields a solution of

$$t^2 - Du^2 = \sigma^2, \qquad \sigma = 1 \text{ or } 2,$$

by means of

$$\alpha = \frac{t - bu}{\sigma}, \quad \beta = -\frac{cu}{\sigma},$$

$$\gamma = \frac{au}{\sigma}, \qquad \delta = \frac{t + bu}{\sigma}.$$

The form need not belong to the class of $(1, 0, -D)$ (Dirichlet's *Vorlesungen*, p. 148).

The author puts

$$\omega = a_0 - \frac{1}{\omega_1}, \quad \omega_1 = a_1 - \frac{1}{\omega_2}, \ \ldots\ldots \ \omega_{r-1} = a_{r-1} - \frac{1}{\omega_r}, \text{ \&c.}$$

If $\omega, \omega_1, \omega_2 \ldots$ are > 2, not having reference to sign, the correspond-

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* The author finds the condition $d_n (y - y_0) < x + y\sqrt{A}$ in order that $\frac{x}{y}$ may be an approximate fraction to $\sqrt{A}$, and argues (p. 28),—"Da nun $y - y_0$ positiv und kleiner als y ist, so ist, wenn $d_n < \sqrt{A}$, $d_n (y - y_0) < y\sqrt{A}$, und um so mehr $d_n (y - y_0) < x + y\sqrt{A}$, also ist auch $\frac{x}{y}$ ein Näherungswerth von $\sqrt{A}$." But, since $x - y\sqrt{A}$ is positive, x is $> y\sqrt{A}$, and $d_n (y - y_0)$ is to be less than $2y\sqrt{A} +$.

ing continued fraction

$$a_0 - \frac{1}{a_1} - \frac{1}{a_2} - \ldots - \frac{1}{a_{r-1}} - \frac{1}{a_r},$$

is called "regular" (*regelmässige*).

Now to (a, b, a_1) corresponds a quadratic $a + bx + a_1x^2 = 0$, and the roots $\frac{-b-\sqrt{D}}{c}$, $\frac{-b+\sqrt{D}}{c}$ are termed respectively the first and second roots of the form.

If $\delta_0, \delta_1 \ldots$ are the nearest integers to $\omega, \omega_1 \ldots$, the respective first roots of the associated forms

$$(a, b, a_1)\ (a_1, b_1, a_2) \ldots (a_r, b_r, a_{r+1}),$$

or, say, $f, f_1 \ldots f_r$, then

$$\begin{aligned} b + b_1 &= -\delta\ a_1, & b_1^2 - a_1a_2 &= 0, \\ b_1 + b_2 &= -\delta_1 a_2, & b_2^2 - a_2a_3 &= 0, \\ \cdots \quad &\cdots \quad \cdots \quad \cdots & \cdots \quad &\cdots \quad \cdots \\ b_{r-1} + b_r &= -\delta_r a_{r+1}, & b_r^2 - a_r a_{r+1} &= D, \end{aligned}$$

and $\omega_1, \omega_2 \ldots$ are > 2, irrespectively of sign.

If the development is continued, we shall arrive at a form f_r, in which, irrespectively of sign, $a_{r+1} \gtreqless a_r$. For, otherwise, we should have $a_1 > a_2 > a_3 > \ldots$ ad inf., which cannot be.

But forms so specialised are finite in number, being subject to the double inequality $\frac{3}{5}\sqrt{D} < b_r > \frac{5}{3}\sqrt{D}$.

From this follows at once the periodicity of the continued fraction, representing the first root of (a, b, a_1). All this is closely analogous to the usual process.

The forms from f_r to its next recurrence are called "reduced forms" by analogy. They are fully characterised by the fact that they give rise to purely periodic regular continued fractions.

The author shows further that, if we start with a so-called "reduced form," all the solutions of $t^2 - Du^2 = \sigma^2$ are obtainable, or, what is the same thing, we get the least solution by means of the first period.

The proof (pp. 627—652) commences by taking a specialised "reduced form" (a, b, c), in which c is negative and absolutely $\gtreqless a$. The cases have to be separately considered in which b is $< \sqrt{D}$ and b is $> \sqrt{D}$; so also the cases in which $a = 0$ or $\frac{\gamma}{\alpha}$ is integer in the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$, which transforms (a, b, c) into itself.

Finally, it is shown that any "reduced form" may be taken with similar result.

The general proposition would, I think, be more readily established

by comparing the "regular" continued fraction with its equivalent, having positive numerators and positive denominators, which I have called the normal continued fraction.

Addition II.

The statements in § 7 require amplification.

Any form belonging to the development of $\sqrt{D}$ in a continued fraction by means of integer limits is properly or improperly equivalent to the original form $q^r - Dr^r$. Let (a, b, c) be such a form; then (1) if we express the major positive root by means of inferior limits as a continued fraction, we shall, according to the usual theory, obtain the solution of the Pellian equation. The forms of the period will alternately have negative middle terms. If the signs of these are changed, the forms become reduced. (2) If we proceed by nearest limits, the development will, in accordance with the results of § 4, be definite in relation to the normal positive development, and in such a way that we must arrive at a leading coefficient = unity. (3) Similarly, if we proceed by inferior and superior limits alternately, we have

$$a-\frac{1}{b}+\frac{1}{c}-\frac{1}{d}+\frac{1}{R} = a-1+\frac{1}{1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{1}+\frac{1}{d-1}+\frac{1}{R}.$$

Two successive letters will not represent unity. In fact, the letters with positive numerators, in the left-hand expression cannot be less than 2.

If $b = 1$, the result is

$$a-1+\frac{1}{c}+\frac{1}{1}+\frac{1}{d-1}+\frac{1}{R}.$$

If also $d = 1$, then it is

$$a-1+\frac{1}{c}+\frac{1}{1+R}.$$

When therefore a partial quotient a occurs in the normal development, there will be one $= a$ or $a+1$ in the development by alternate limits, and the form to which this partial quotient belongs will have unity for a leading coefficient when a is twice the greatest integer in $\sqrt{D}$. (4) Suppose, again, we proceed arbitrarily as to limits, and obtain the form (a, b, c) where a is positive. If now we go on

with superior limits, the normal fraction being

$$a+\frac{1}{a_1}+\frac{1}{a_2}+\ldots,$$

our result will be of the form

$$[a+1,\ \overline{a_1-1},\ a_2+2,\ \overline{a_3-1}\ \ldots],$$

and we can apply the same reasoning as in Addition I. (1).

(5) We may proceed arbitrarily as to limits, except the condition that, when the irrational is <2, the inferior limit must be taken. For in this case no partial quotient $=2$ is immediately followed by a negative numerator. The transformation of such a continued fraction into the normal fraction equivalent to it is similar to that of the continued fraction obtained by taking the nearest limits, and no zero denominator is introduced except in the case of three successive terms of the form

$$\frac{1}{a}-\frac{1}{1}+\frac{1}{R}=\frac{1}{a}-1+\frac{1}{1}+\frac{1}{0}+\frac{1}{R}=\frac{1}{a-1}+\frac{1}{1+R}.$$

(6) There is another case worth noting, and with more detail. We may proceed by taking as limits the integers next to, but furthest from, the true values of the irrationals.

Every partial quotient is then 1 or 2. When the partial quotient is unity, it is always immediately followed by a positive unit numerator; when it is 2, by a negative unit numerator.

Suppose the negative continued fraction (to use Herr Stern's expression) $[a+1,\ \overline{a_1-1},\ a_2+2,\ \overline{a_3-1},\ \ldots]$ .....(a)

is given. Applying repeatedly the identity

$$\frac{1}{K}+\frac{1}{R}=1-\frac{1}{1}+\frac{1}{K-1}+\frac{1}{R}$$

to the terms

$$-\frac{1}{a_{2m}+2}-\frac{1}{R},$$

we get

$$-\left(1-\frac{1}{2}-\frac{1}{2}-\ldots-\frac{1}{2}-\frac{1}{1}+\frac{1}{2}-\frac{1}{R}\right),$$

the denominator 2 being repeated $a_{2m}-2$ times.

Hence the fraction (a) is equivalent to

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+,\ \overline{a_2-2},\ 1,\ 2^+,\ \overline{a_3-2},\ 1,\ 2^+,\ \ldots]\ .....(b),$$

where 2^+ is written to indicate that the corresponding numerator is positive.

We will suppose at present that no a is <2. The above fraction fulfils the conditions required.

Now any form (A, B, C) is identical with the form

$$\left(\frac{(A+B)^2-D}{m}, \frac{(A+B)(B+C)-D}{m}, \frac{(B+C)^2-D}{m}\right),$$

where m is written for $A+2B+C$, and D is the determinant.

Consider the form into which (A, B, C) is changed by the substitution $\begin{pmatrix}\lambda, & -1\\ 1, & 0\end{pmatrix}$.

The resulting form will have for its leading coefficient $A\lambda^2+2B\lambda+C$, and the sum of the coefficients is $A(\lambda-1)^2+2B(\lambda-1)+C$, which would have been the leading coefficient, if we had transformed by the substitution $\begin{pmatrix}\lambda, & +1\\ 1, & 0\end{pmatrix}$.

These leading coefficients will be of opposite signs when λ is the superior limit and consequently $\lambda-1$ the inferior limit. But, having regard to the genesis of the fraction (a), one sees that the fraction

$$[a+1,\ \overline{a_1-1},\ a_2+2\ \ldots\ a_{2m}+1]$$

is equal to the fraction

$$(a,\ a_1,\ a_2\ \ldots\ a_{2m}).$$

Also, if $\lambda = 2$, the sum of the coefficients of the derived form is the same as that of the coefficients of the form from which it is derived.

It follows, then, that the sum of the coefficients of any form belonging to (a) is the same as the leading term of the corresponding form of the positive normal fraction, and that, as often as λ is successively $= 2$, the sum of the coefficients is constant.

Moreover, since the leading terms of the forms belonging to (a) are positive, the sums of the coefficients of the forms are negative, and equal respectively to the negative leading coefficients of the forms of the equivalent normal continued fraction representing $\sqrt{D}$.

We have seen that, whenever $q^2-Dr^2=k$ is possible, k being a positive integer $<2\sqrt{D}$, then k will appear as a leading term of a form belonging to the continued fraction (a); *à fortiori*, all the positive leading terms of forms belonging to the continued fraction with positive unit numerators, and representing $\sqrt{D}$, will appear among the leading terms of forms belonging to the negative continued fraction representing $\sqrt{D}$. These leading terms occur when we stop at the last partial quotient in a series of the value 2; for

$$\begin{aligned}&[a+1,\ \overline{a_1-1},\ a_2+2,\ \ldots\ \overline{a_{2m+1}-1}]\\ &= (a,\quad a_1,\quad a_2,\quad \ldots\ a_{2m+1}).\end{aligned}$$

The same reasoning precisely may be applied to the continued fraction obtained by taking first of all the inferior limit, and then proceeding exclusively by superior limits.

We have a fraction

$$[a,\ a_1^+ +1,\ \overline{a_2-1},\ \overline{a_3+2},\ \overline{a_4-1},\ \ldots] \ldots\ldots\ldots\ldots\ldots\ldots (c),$$

and it will be seen that the sums of the coefficients of all the forms belonging to it are equal to the corresponding positive leading terms of the normal forms, while all the negative leading terms of the normal forms are found among the leading terms of the forms belonging to (c). These are given by the fractions

$$[a,\ a_1^+ +1,\ \overline{a_2-1},\ \overline{a_3+2}\ \ldots\ \overline{a_{2m}-1}].$$

In the same way as (a), we can transform (c) into (b).

Still, supposing that no a is <2, we can now see without much difficulty that every fraction

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+\ \ldots\ \overline{a_{2m+1}-2},\ 1,\ 2^+]$$
$$= [a+1,\ \overline{a_1-1},\ a_2+2\ \ldots\ \overline{a_{2m+1}-1}],$$

and every fraction

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+\ \ldots\ \overline{a_{2m}-2},\ 1,\ 2^+]$$
$$= [a,\ a_1^+ +1,\ \overline{a_2-1},\ a_3+2\ \ldots\ \overline{a_{2m}-1}].$$

Also as to fractions terminating with an element $-\frac{1}{2}$, each one, derived from (b), has one equivalent fraction terminating with an element $-\frac{1}{2}$ derived from (a).

But there remains a species of fraction derived from (a) (and having the last of a series of elements $-\frac{1}{2}$ for its final term), which is equivalent to a fraction derived from (b) of the form

$$[a+1,\ \overline{a_1-2},\ 1,\ 2^+ \ldots 1,\ 2^+,\ \overline{a_k-2},\ 1].$$

For
$$-\frac{1}{1}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\ldots-\frac{1}{2}-\frac{1}{1}=-\frac{1}{2}.$$

The result is similar when we compare (b) with (c), and we conclude generally that all the leading terms of forms belonging to (a) and (c) reappear as leading terms of forms belonging to (b), and no others.

Moreover, the sums of the coefficients of the forms, excluding the first belonging to (b) are equal to the leading terms of the forms belonging to the normal positive continued fraction representing $\sqrt{D}$.

But the case of $a_k = 1$ remains to be considered. If $a_k = 2$, $\overline{a_k-2}$ simply means that there are no elements $-\frac{1}{2}$ corresponding to $\overline{a_k-2}$,

and we pass at once to $-\frac{1}{1}$. When $a_k = 1$, the matter is of more consequence. Yet we only have to leave these terms $\frac{1}{1}$ as they are. Thus $(a, a_1, a_2, 1, 1, 1 \ldots 1, a_3, a_4 \ldots)$, when 1 is k times repeated in succession, becomes

$$(a+1,\ \overline{a_1-2},\ 1,\ 2^+,\ \overline{a_2-2},\ 1,\ 1^+,\ 1^+,\ 1^+ \ldots$$
$$\ldots 2^+,\ 2,\ \overline{a_3-2},\ 1,\ 2^+,\ a_4-2,\ 1,\ 2^+ \ldots),$$

where 1^+ is also k times repeated.

If, however, a series of units begins directly after a, the transformation is $(a,\ 1^+,\ 1^+ \ldots 1^+,\ 2^+,\ \overline{a_1-2},\ 1,\ 2^+ \ldots)$,

or $(a,\ 1^+,\ 1^+ \ldots 1^+,\ 2^+,\ \overline{a_1-2},\ 1,\ 1^+ \ldots)$,

if $\frac{1}{a_1}$ is followed by terms $\frac{1}{1}$ immediately.

Now these changes do not affect the fact that all the leading terms of the forms belonging to the two negative continued fractions reappear as leading terms in the forms belonging to the continued fraction obtained by taking as limits the integers next to, but furthest from, the values of the irrationals.

This being so, the conclusion is reached, that such a development gives, when possible, the solution of $q^2 - Dr^2 = \pm k$, when k is a positive integer $< 2\sqrt{D}$.

As an example take $D = 19$; the normal development is

$$(4,\ 2,\ 1,\ 3,\ 1,\ 2,\ 8,\ 2 \ldots),$$

so that we have partial quotients $= 1$ and 2. Paying attention to this, we have for the three negative developments

$$[5,\ 2,\ 3,\ 2,\ 2,\ 3,\ 2,\ 10,\ 2 \ldots],$$
$$[4,\ 3^+,\ 5,\ 4,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 4,\ 5 \ldots],$$
$$[5,\ 1^+,\ 1^+,\ 2^+,\ 2,\ 1,\ 1^+,\ 2^+,\ 1,\ 2^+,\ \overline{8-2},\ 1,\ 2^+],$$

and the approximations derived from the third development are those of the first two together. In obtaining such developments, care must be taken to adhere to the positive root when the two first coefficients of the form operated on are of the same sign. *E.g.*, in the case of $-2q^2 - 2qr + 9r^2$ the transformation is obtained from

$$\frac{1-\sqrt{19}}{-2}.$$

Thursday, June 12th, 1884.

Prof. HENRICI, F.R.S., President, in the Chair.

Mr. G. S. Ely, Ph.D., Professor of Mathematics in Buchtel College, Akron, Ohio, was elected a Member of the Society.

The President announced that the Council had awarded the De Morgan Medal to Prof. Cayley, F.R.S.

Mr. Tucker communicated an abstract of a paper by Prof. H. Lamb, entitled "Note on the Induction of Electric Currents in a Cylinder placed across the Lines of Magnetic Force."

Mr. J. Hammond gave some results of a paper by him in the *American Journal of Mathematics.*

The following presents were received :—

"Proceedings of the Royal Society," Vol. xxxvi., No. 230.

"Proceedings of the Physical Society of London," Vol. v., Pt. 5, Oct. 1883, March 1884 ; Vol. vi., Pt. 1, April—May, 1884.

"Educational Times," for June.

"Memoirs of the Manchester Literary and Philosophical Society," 3rd Series, Vol. vii. and ix.

"Proceedings of the Manchester Literary and Philosophical Society," Vols. xx., xxi., and xxii.

"On Electrical Motions in a Spherical Conductor," by Prof. H. Lamb (from "Philosophical Transactions," Pt. ii., 1883).

"Johns Hopkins University Circulars," Vol. iii., No. 30.

"Proceedings of the Canadian Institute," Vol. ii., Fasc. No. 1, 8vo ; Toronto, 1884.

"On an Unsymmetrical Law of Error in the Position of a Point in Space," by E. L. De Forest, (from "Transactions of Connecticut Academy").

"Atti del R. Istituto Veneto," Tomo i., Ser. vi., Disp. 4, 5, 6, 7, 8, 9, 10 ; Tomo ii., Ser. vi., Disp. 1, 2.

"Bulletin de la Société Mathématique de France," T. xii., No. 1.

"Bulletin des Sciences Mathématiques et Physiques," 2nd series, T. vii. (Index) ; T. viii., April and May.

"Beiblätter zu den Annalen der Physik und Chemie," B. viii., St. 4 and 5.

"Acta Mathematica," iv. 3.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," Nos. 1—17.

Note on the Induction of Electric Currents in a Cylinder placed across the Lines of Magnetic Force. By Horace Lamb, M.A.

[*Read June 12th*, 1884.]

This note may be regarded as supplementary to a previous communication "On the Induction of Electric Currents in Cylindrical and Spherical Conductors."* I there discussed the currents induced in a long circular cylinder by variation of the intensity of a uniform magnetic field, the cylinder being supposed placed with its length parallel to the lines of force. I now consider the case where the cylinder is placed transverse to the lines of force, which are assumed to be all parallel to one plane. If this condition be fulfilled, it is unnecessary further to restrict the problem by assuming the field to be uniform. For mathematical simplicity, we suppose the cylinder to be infinitely long; it will be evident, however, that our results are applicable without sensible modification to the case of a finite cylinder whose length is a moderate multiple of its diameter. By combining the results of the former paper with those obtained below, we can solve the case of a cylinder placed obliquely in a uniform field of force.

Let the axis of the cylinder be taken as axis of z. If a, b, c denote, as usual, the components of magnetic induction, the condition above stated requires

$$c = 0, \quad \frac{da}{dz} = 0, \quad \frac{db}{dz} = 0,$$

everywhere. Hence, if u, v, w be the components of current in the cylinder

$$u = 0, \quad v = 0, \quad \frac{db}{dx} - \frac{da}{dy} = 4\pi\mu w \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where μ is the magnetic permeability of the substance. If F, G, H denote the components of electric momentum, we shall then have

$$F = 0, \quad G = 0,$$

$$a = \frac{dH}{dy}, \quad b = -\frac{dH}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Since there is no accumulation of free electricity anywhere, the "electric potential" ϕ is zero, and we have

$$\rho w = -\frac{dH}{dt} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

* Cf. *supra*, pp. 139 and 149.

where ρ is the specific resistance of the substance. From (1), (2), and (3), we obtain

$$\frac{d^2H}{dx^2}+\frac{d^2H}{dy^2}=\frac{4\pi\mu}{\rho}\frac{dH}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

In the space surrounding the conductor, we shall have

$$\frac{d^2H}{dx^2}+\frac{d^2H}{dy^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The conditions to be satisfied at the boundary of a conductor are as follows. In the first place the components of electric momentum must be continuous. This implies the continuity of the normal component of magnetic induction. Again, the tangential components of magnetic induction, just inside and just outside the surface, must agree in direction and be in the ratio of μ to 1. Hence if the accents $'$ and $''$ refer to the inside and the outside of the surface respectively, we have in our case

$$H'=H'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and

$$a'\sin\theta-b'\cos\theta=\mu\,(a''\sin\theta-b''\cos\theta),$$

or

$$\frac{dH'}{dr}=\mu\frac{dH''}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where r, θ are polar coordinates in the plane xy.

If we introduce the assumption that $H\propto e^{\lambda t}$, the equation (4) becomes

$$\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+k^2\right)H=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

provided

$$k^2=-4\pi\lambda\mu/\rho \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

A particular solution of (8), subject to the condition of finiteness, when $r=0$, is

$$H=A_n\sin n\theta\,.\,J_n(kr) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10),$$

where n is a positive integer, and J_n denotes the Bessel's function of the nth order. The corresponding value of H for the space outside the cylinder is of the form

$$H=(C_nr^n+D_nr^{-n})\sin n\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

To obtain the general solution, we have only to add to the above similar terms in which $\sin n\theta$ is replaced by $\cos n\theta$, and prefix the sign Σ of summation with respect to n.

The surface-conditions (6) and (7) give

$$\left.\begin{aligned}A_nJ_n(kR)&=C_nR^n+D_nR^{-n}\\ A_nkRJ_n'(kR)&=n\mu\,(C_nR^n-D_nR^{-n})\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12),$$

where R is the radius of the cylinder.

In the case of *free* currents, we have $C_n=0$, and thence, from (12),

$$kRJ_n'(kR)+n\mu J_n(kR)=0,$$

which may also be written

$$kRJ_{n-1}(kR)+n(\mu-1)J_n(kR)=0 \quad \ldots\ldots(13).$$

When the roots of this equation in kR are known, the corresponding values of the modulus of decay are given by

$$-\lambda^{-1}=\frac{4\pi\mu R^2}{\rho}(kR)^{-2} \quad \ldots\ldots(14).$$

The distribution of current in the cylinder is given by

$$w=\frac{k^2}{4\pi\mu}A_nJ_n(kr)\sin n\theta \quad \ldots\ldots(15).$$

The most interesting type is that corresponding to the lowest root of (13) in the case $n=1$.* The direction of the current then changes sign as we cross the plane zx. In the case of a finite cylinder, each pair of current lines symmetrically situated on opposite sides of this plane would be replaced by a sort of elongated oval.

For a non-magnetic substance $\mu=1$, and (13) reduces to

$$J_{n-1}(kR)=0 \quad \ldots\ldots(16).$$

Hence the modulus of decay, in the case just referred to, has the same value as in the most persistent type of free transverse currents, where the currents flow in circles round the axis of the cylinder.

When, on the other hand, μ is large, we have as a first approximation

$$J_n(kR)=0 \quad \ldots\ldots(17),$$

and as a second $\quad kR=\left\{1-\frac{1}{n(\mu-1)}\right\}\vartheta,$

where ϑ is a root of (17). It appears that the lines of magnetization, which are given by $H=$ const., form for the most part closed curves. If we take $\mu=400$, which corresponds roughly to the case of iron, the lowest root of (13) in the case $n=1$ is $kR=3{\cdot}822$. Hence the modulus of decay is less than in the most persistent type of free transverse currents (for which $kR=2{\cdot}404$) in the ratio $1:2{\cdot}53$.

As in the former paper, we can readily investigate the currents induced by the sudden cessation of a magnetic field of uniform intensity I (parallel to x, say). At the time $t=0$, we then have, inside the cylinder,

$$H=\frac{2\mu I}{\mu+1}r\sin\theta \quad \ldots\ldots(18).$$

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* The case $n=0$ is not included in the preceding formulæ. It is easily seen that there are possible arrangements of free currents, in which

$$H=A_0J_0(kr),$$

the values of k being determined by

$$J_1(kR)=0.$$

The integral current across the section of the cylinder is zero, and there is no magnetic effect outside.

At any subsequent time t, we shall have, by our previous formulæ,

$$H = \Sigma A_p J_1(k_p r) \sin\theta \,.\, e^{\lambda_p t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19),$$

where k_p denotes the pth root* of

$$kRJ_1'(kR) + \mu J_1(kR) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (20),$$

and the corresponding value of λ is given by (9). Hence we must have

$$\Sigma A_p J_1(k_p r) = \frac{2\mu I}{\mu+1} r \quad \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

from $r = 0$ to $r = R$. If k_p, k_q be different roots of (13), we find by the usual methods (see Rayleigh's *Sound*, §§ 203, 332)

$$\int_0^R J_1(k_p r) \,.\, J_1(k_q r)\, r\, dr = 0,$$

$$\int_0^R J_1^2(k_p r)\, r\, dr = \frac{k_p^2 R^2 + \mu^2 - 1}{2k_p^2} J_1^2(k_p R),$$

$$\int_0^R J_1(k_p r)\, r^2 dr = \frac{R^2}{k_p} J_2(k_p R) = (\mu+1)\frac{R}{k_p^2} J_1(k_p R).$$

Hence

$$A_p = \frac{4\mu R I}{k_p^2 R^2 + \mu^2 - 1} \cdot \frac{1}{J_1(k_p R)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

The value of H outside the cylinder is given by

$$H = \Sigma D_p \frac{\sin\theta}{r} e^{\lambda_p t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (23),$$

where

$$D_p = RJ_1(k_p R)\, A_p \quad \ldots\ldots\ldots\ldots\ldots\ldots (24).$$

Hence at the time t the external effect is the same as if the cylinder were uniformly magnetized transversely with a magnetic moment per unit length

$$= \tfrac{1}{2}\Sigma D_p e^{\lambda_p t}$$

$$= 2\mu R^2 I \,.\, \Sigma \frac{e^{-\zeta^2 t/\tau}}{\zeta^2 + \mu^2 - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

where $\zeta_p = k_p R$, and $\tau = 4\pi\mu R^2/\rho$. The electro-kinetic energy per unit length is

$$\tfrac{1}{2}\int_0^{2\pi}\int_0^R Hw\, r\, d\theta\, dr = \mu R^2 I^2 \,.\, \Sigma \frac{e^{-2\zeta^2 t/\tau}}{\zeta^2 + \mu^2 - 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (26).$$

We may prove, by the method of Rayleigh's *Sound*, § 332, that

$$\Sigma \frac{1}{\zeta^2 + \mu^2 - 1} = \frac{1}{2(\mu+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (27),\dagger$$

---

* Not counting the zero root.

† More generally, if $\zeta_1, \zeta_2, \ldots$ be the finite roots of

$$\zeta J_n'(\zeta) + \beta J_n(\zeta) = 0,$$

where β is any constant, then

$$\Sigma \frac{1}{\zeta^2 + \beta^2 - n^2} = \frac{1}{2(\beta+n)}.$$

so that (26) becomes, for $t = 0$,

$$\frac{\mu R^2 I^2}{2(\mu+1)}.$$

This can be easily verified from the consideration that initially there is no current except at the surface of the cylinder, where we have a current-sheet whose strength w' is given by

$$w' = \frac{I}{2\pi} \sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28).$$

In treating the general problem of induced currents, subject to the restriction stated at the beginning of this note, we must suppose the disturbance in the field expressed, as regards the time, by a series of simple harmonic terms. If p be the frequency of any one of these, we have $\lambda = 2\pi i p$, and the corresponding value of k is then given by (9). The coefficients C_n in (11) are known when the nature of the disturbance in the field is given, and the values of A_n and D_n are then to be found from (12); viz., we have

$$\left.\begin{aligned} A_n &= \frac{2n\mu R^n}{kRJ_{n-1}(kR) + n(\mu-1)J_n(kR)} C_n \\ D_n &= -\frac{kRJ_{n-1}(kR) - n(\mu+1)J_n(kR)}{kRJ_{n-1}(kR) + n(\mu-1)J_n(kR)} R^{2n} C_n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (29).$$

When $\mu = 1$, these take the forms

$$\left.\begin{aligned} A_n &= \frac{2nR^n}{kRJ_{n-1}(kR)} C_n \\ D_n &= \frac{J_{n+1}(kR)}{J_{n-1}(kR)} R^{2n} C_n \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The same relations hold between the corresponding coefficients of $\cos n\theta$.

The interpretation of these results, in the two extreme cases where kR is very large or very small, presents no difficulty, and in this way we might examine the effect of the frequency of the disturbance on the distribution of current within the cylinder. This point has, however, been sufficiently illustrated in former papers.

The following presents were received in the Recess :—

"Proceedings of the Royal Society," Vol. xxxvi., Nos. 231 and 232.

"Educational Times," for July, August, September, and October.

"Mathematical Questions, with their Solutions, from the Educational Times," Vol. xli., 8vo; London, 1884.

"Proceedings of the Physical Society of London," Vol. vi., Part ii., June to August, 1884.

"Greek Geometry, from Thales to Euclid," Part iii., by G. J. Allman, 8vo; Dublin, 1884.

"Proceedings of the Royal Society of Edinburgh," Vol. xi., No. 110, 1881-82, Vol. xii., No. 113, 1882-83.

"Proceedings of the Royal Irish Academy," Polite Literature—Ser. ii., Vol. ii., No. 5, 1884; Science—Ser. ii., Vol. iv., Nos. 1 and 2, 1884.

"Transactions of the Royal Irish Academy—Science," Vol. xxviii., Nos. 14, 15, and 16.

"Johns Hopkins University Circulars," Vol. iii., No. 32.

"Observations of the Great Comet of 1882, made at the U.S. Naval Observatory," 4to; Washington, 1883.

"Proceedings of the Canadian Institute," Toronto, Vol. ii., No. 2, 8vo; Toronto, 1884.

"Acta Mathematica," Vol. iv., Nos. 1, 2, and 4.

"Crelle," Bd. xcvi., H. 3 and 4; Bd. xcvii., H. 1, 2.

"Atti della R. Accademia dei Lincei," Vol. viii., F. 11, 12, 13, 14, 15.

"Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin," xviii. to xxxix.

"Archiv for Mathematik og Naturvidenskab," 8 B., 1, 2, 3, 4 Hefte; 9 B., 1, 2, 3 H.

"Beiblätter zu den Annalen der Physik und Chemie," B. viii., St. 6, 7, 8, 9.

"Bulletin des Sciences Mathématiques," Série 2, Tome viii., June, July, August, and September, 1884.

"Bulletin de la Société Mathématique de France," T. xii., Nos. 2 and 3.

"Archives Néerlandaises des Sciences Exactes et Naturelles," T. xix., No. 2.

"Jornal de la Sciencias Mathematicas e Astronomicas," Vol. v., No. 4.

"Reale Istituto Lombardo—Rendiconti," Ser. ii., Vol. xv.

"Memorie del Reale Istituto Lombardo," Vol. xv., xvi. della Serie iii., Fasc. 1.

"Memorie della R. Accademia, &c. in Modena," Serie ii., Vol. ii.; Modena, 1884.

Benedictus de Spinoza—"Stelkonstige Reeckening van den Regenboog," and "Reeckening van Kanssen," (Two nearly unknown treatises,) Reprint, small 4to.

Simon Stevin—"Vande Spiegeling der Singkonst," et "Vande Molens," Deux traités inédits; Amsterdam, 1884, Reprint, small 4to.

Albert Girard—"Invention Nouvelle en l'Algèbre," Reprint, small 4to; Leiden, 1884.

"Théorie des Approximations Numériques—Notions de Calcul Approximatif," par Ch. Galopin-Schaub; Genève, 1884.

The following were presented by Mr. Tucker:—

"A Treatise on the Stability of a given State of Motion, particularly Steady Motion" (Adams Prize Essay, 1877), by E. J. Routh, F.R.S.

"An Elementary Treatise on Solid Geometry," by Charles Smith, M.A.; 1884.

"A Collection of Examples on the Analytic Geometry of Plane Conics," to which are added some Examples on Sphero-Conics, by Ralph A. Roberts, M.A.; 1884.

APPENDIX.

Dr. Taylor, Master of St. John's College, Cambridge, in a paper—an abstract of which was printed in the *Proceedings*, Vol. vi., pp. 123 to 125—called a certain circle employed in the Geometry of Plane Conics "Walker's Circle." He calls the attention of members to the fact, that he afterwards traced the circle further back, and now attributes its discovery to Boscovich. (See *An Introduction to the Ancient and Modern Geometry of Conics*, Prolegomena, p. lxxii., 1881.)*

The brief note on "Theory of Orthoptic Loci" (*Proceedings*, Vol. xiii., p. 179), by the same gentleman, has been recast and published as a "Note of a Theory of Orthoptic and Isoptic Loci" in the *Proceedings of the Royal Society* (No. 232, 1884).

The following simple construction in elucidation of (*a*), p. 66, *supra*, is sent us by the author.

Take QPM an ordinate cutting the ellipse in P and the auxiliary circle in Q: join Q to the centre C. Draw CR making the same angle with the major axis CA as CQ does, and take $CR = a - b$ (*i.e.*, difference between semi-axes), then PR is the normal at P to the ellipse.

Mr. Griffiths also sends a "Historical Note on the Transformation of Elliptic Functions."

"The idea of composition does not seem to have occurred to Jacobi, though in a letter to Legendre, dated 5th August, 1827, he mentions, as I have recently discovered, the following theorem, viz.,

"'Soit $\lambda^2 + \lambda'^2 = 1$, soit en général ψ^m un angle tel, que

$$F(\lambda', \psi^m) = \frac{m}{p} F'(\lambda'),$$

qu'on fasse

$$\operatorname{tg}\theta' = \operatorname{tg}\psi \operatorname{cosec}\psi', \quad \operatorname{tg}\theta''' = \operatorname{tg}\psi \operatorname{cosec}\psi''', \ldots \operatorname{tg}\theta^{p-2} = \operatorname{tg}\psi \operatorname{cosec}\psi^{p-2},$$

soit enfin $$\theta = 2(\theta' - \theta''' + \theta^{v} - \ldots \mp \theta^{p-2} \pm \tfrac{1}{2}\psi),$$

on aura $F(k, \theta) = MF(\lambda, \psi)$.' (See Jacobi's works, Berlin, 1881.)

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* Dr. Taylor, in a foot-note, writes:—"An edition of Boscovich's *Elementa* seems to have been published at Rome in his name in 1752-3; but I have only seen the 'Editio Prima Veneta.'" We have not seen an edition of the date he gives, but we have collated with the "Veneta" an edition "Romæ MDCCLIV., pp. xxvi. + 468." The two editions are very diverse in all particulars except in the contents, which, if our memory serves us, are identical.

"In a reply (14th April, 1828), Legendre remarks 'qu'en faisant $\gamma' = \cot^2\psi'$, $\gamma''' = \cot^2\psi'''$..., $x = \sin\psi$, $y = \sin\theta$, on a

$$y = \frac{\mu x\left(1+\frac{\lambda^2x^2}{\gamma'}\right)\left(1+\frac{\lambda^2x^2}{\gamma'''}\right)\ldots}{(1+\gamma'x^2)(1+\gamma'''x^2)\ldots},$$

But neither of them notices the fact that $z = \cos 2\theta' = \frac{1-(1+\alpha'^2)\,x^2}{1-\alpha^2x^2}$, where $\alpha^2+\alpha'^2 = 1$, is a quadric transformation, and so no advance as regards a theory of composition is made beyond the mention of the formula in question.

"I can find no trace in Jacobi's writings of any idea bearing on the subject of composition of linear forms.

"An attempt at constructing a theory based upon the latter kind of composition has recently been made by the present writer."

The short note, "Symmedians and the Triplicate-Ratio Circle," has been published in the *Educational Times*, January, 1884 (*Reprint* from the *Educational Times*, Vol. xli., p. 26).

The "Note on Inverse-Coordinate Curves" (communicated February 9th, 1882, *Proceedings*, Vol. xiii., p. 85) appeared in the *Educational Times*, March, 1884 (*Reprint*, Vol. xli., p. 56).

Mr. Buchheim's "An Extension of Pascal's Theorem to Space of Three Dimensions" (p. 82) may be read in the *Messenger of Mathematics*, No. clxi., pp. 74, 75 (September, 1884).

Mr. Fortey's paper "On Contacts and Isolations" (pp. 98—102) was illustrated by the following examples:—

(*a*) Find the number of permutations of 9 white, 7 black, and 5 red balls, taken altogether, in which there are exactly 5 contacts of the white and 4 of the black. (*Answer*, 105840.)

(*b*) In how many ways may 5 Englishmen, 4 Frenchmen, 3 Germans, and 3 Italians be made to stand in a row, so that the French, Germans, and Italians may be isolated? (*Answer*, 2012388.)

The paper "On a Group of Circles," &c., is published in the *Quarterly Journal of Mathematics* (Vol. xx., No. 77).

Also Mr. Tucker's note on "Some Properties of Two Lines in the Plane of a Triangle" (p. 218) occurs in the same Journal (Vol. xx., No. 78; *cf.* also *Educational Times*, July, 1884, *Reprint*, Vol. xlii., p. 25).

Professor Cremona's communication, referred to in the foot-note, p. 242, is entitled "On a Geometrical Transformation of the Fourth Order, in Space of Three Dimensions, the Inverse Transformation being of the Sixth Order." [*Trans. of the Royal Irish Academy*, Vol. xxviii. *Science*, Vol. xvi., pp. 279—284 (July, 1884).]

For the remarks referred to on p. 103, reference may be made to Professor Sylvester's *Lectures on the Principles of Universal Algebra* (*American Journal of Mathematics*, Vol. vi., No. 3, pp. 270—286), and to a communication made by him to *Nature* (Vol. 31, p. 35, Nov. 13, 1884).

The following notes bearing upon the subject of pp. 316—321, of the *Appendix*, Vol. xiv., may be of interest to some members:—

M. E. Lemoine has forwarded copies of his papers, so that we are now able to give their exact titles (*cf.* p. 320, l. c.). That read at Lyons (August 27, 1873) is entitled "Sur quelques propriétés d'un point remarquable d'un triangle"; the point is what we call the Symmedian-point, and we first meet here with the distinguishing property of a circle connected with the point which led us to call it the Triplicate-Ratio Circle. That read at Lille (August 22, 1874) is called "Note sur les propriétés du centre des Médianes antiparallèles dans un triangle" (*Suite*): this is the point above referred to.

At the meeting of the *Association Française pour l'Avancement des Sciences* in 1883 (*Congrès de Rouen*), M. Lemoine communicated the following Note (August 18th):—"Sur les quatre groupes de deux points d'un triangle ABC qui sont en même temps les foyers d'une conique inscrite et d'une conique circonscrite à ce triangle." The centre of the two homofocal ellipses is shown to be the Symmedian-point of the triangle $O_aO_bO_c$ (where O_a, &c. are the ex-centres of ABC). Several neat properties are given, and reference frequently made to a paper read in the previous year at the Congrès de la Rochelle (which we have not seen). The last pamphlet we have from the author (bearing on this special subject) is entitled "Nouveaux points remarquables du plan d'un triangle" (extrait du *Journal de Mathématiques Spéciales*, Paris, 1883). This paper, in 16 pp. octavo, gives a number of problems connected with the above-named point: there is added to it a six-page paper, "Quelques Théorèmes sur les droites menées par un point du plan d'un triangle parallèlement à ses cotés."

Professor J. Neuberg has written a very interesting "Mémoire sur le Tétraèdre." (published in Tome xxxvii. des *Mémoires couronnées et autres Mémoires*, publiés par l'Académie royale de Belgique, Bruxelles, 1884), in which much use is made of the "points et cercles de Lemoine et de Brocard" (*i.e.*, the Symmedian and Brocard points and the "T.R." and Brocard circles). The paper fills up 72 octavo pp.

Dr. Fuhrmann kindly corrects an oversight in our *Appendix* Note (which we had previously detected, and which is readily seen to be such on reference to our "T. R." circle paper).* He sends also the follow-

* See Vol. xiv., p. 318, lines 8 and 9 up, where *dele* "of the circle through the feet . . . . . on the sides of ABC."

ing relation for the Brocard angle ω :

$$4\cot\omega = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} + \cot\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\cot\frac{B}{2}\tan\frac{C}{2}$$

$$+\tan\frac{A}{2}\tan\frac{B}{2}\cot\frac{C}{2}.$$

Professor A. Artzt (*Oberlehrer* of the Gymnasium at Recklinghausen) sends us a copy of his "Untersuchungen über ähnliche Punktreihen auf den Seiten eines Dreiecks und auf deren Mittelsenkrechten, sowie über Kongruente Strahlenbüschel aus den Ecken desselben; ein Beitrag zur Geometrie des Brocards'chen Kreises."*

In the *Zeitschrift f. Mathem. u. Natur. Unterr.*, xv.,† the following mathematicians contribute problems or solutions bearing upon the Brocard Circle : Messrs. Artzt, Fuhrmann, Stoll, Brocard, Stegemann, Adami, Emmerich, and Böcklen‡ (pp. 191—193, 195, 285—288, 290, 353—358, 364—366, 434—436, 521, 522, 524).

To the Meeting of the *Association Française* at Rouen, Captain Brocard (August 23, 1883) communicated a paper entitled "Nouvelles propriétés du triangle" (11 pp.). In this we find further references to articles by M. G. Tarry (*Mathesis*, t. ii., p. 73; *Comptes Rendus*, April 3, 1882), A. Morel (*Journal de Math. Elém.*, t. vii., 1883, pp. 10, 33, 62, 97, &c.). Finally, the "T. R." circle comes in as a particular case in Mr. H. M. Taylor's paper in the present volume (pp. 137–38).

In working further at the subject, we have arrived at the results which follow :—

Take a triangle ABC, $(B>C)$, and let AD, BE, CF be the perpendiculars intersecting in the orthocentre P, and D', E', F' the mid-points of the sides of ABC.

Refer the figure to CD, DA as axes.

The Simson lines for D, E, F with regard to the sides of $D'E'F'$ (since the nine-point circle is the common circum-circle) are given by the equations—

for D, $\quad y - R\sin B\sin C = -x\cot(B-C)$,

for E, $\quad y - R\sin B\sin C = \cot A\,(x - R\cos A\sin 2C)$,

for F, $\quad y - R\sin B\sin C = -\cot A\,(x + R\cos A\sin 2B)$.

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* It occupies 22 pages of the "Programm des Gymnasiums zu Recklinghausen, liv. Schuljahr, 1883-1884, herausgegeben von dem Direktor Dr. B. Hölscher."

† Sometimes styled "*Journal d'Hoffmann*." We are indebted to the editor, Dr. H. Lieber, for the portions of the *Journal* bearing on the present subject.

‡ This mathematician has found that the six circum-circles of AOB, $AO'B$, BOC, $BO'C$, COA, $CO'A$, intersect again in three points A'', B'', C'' on the Brocard-circle : these points G. Tarry has shown to be points of intersection of the B.-circle with the circles HBC, HCA, HAB (p. 355).

These three lines evidently pass through one point (ω) whose co-ordinates are

$$R\cos^2 A\sin(B-C),\quad \frac{R}{2}(\cos A-\cos 2A\cos B-C)\ \ldots\ldots(1).$$

The latter coordinate can be written symmetrically

$$R\,[\sin A\,(\sin A\sin B\sin C)-\cos A\,(\cos A\cos B\cos C)].$$

Now obtain, similarly, the equations of the Simson lines for D', E', F' with regard to the sides of DEF; they are—

for D', $\quad x = R\cos^2 A\sin(B-C)$,

for E', $\quad y\sin C - x\cos C = \frac{R}{2}\sin 2B\cos(A-C)$,

for F', $\quad y\sin B + x\cos B = \frac{R}{2}\sin 2C\cos(A-B)$.

These last three lines also pass through ω.

On referring to equation xv. of "The Triplicate-Ratio Circle" (*Quart. Jour.*, Vol. xix., 76), we find the equation to the line on which the circum-centre, the Brocard (or "T. R.")-centre, and the Symmedian point lie, to be $\quad a\sin(B-C)+\ldots+\ldots=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$.

Hence, from (1) above, we see that ω lies on this line. We see then that the six Simson lines and the circum-Brocard axis pass through a single point, viz. ω.

From E let fall EL perpendicular on DF, then the coordinates of L are $\quad R\cos A\cos 2A\sin 2C,\quad -R\sin A\cos 2A\sin 2C$,

and those of the mid-point of EL (l say) are

$$2R\sin 2C\cos^3 A,\quad 2R\sin 2C\sin^3 A\,;$$

hence equation to Dl is $y = x\tan^3 A$ (*cf. Mess. of Math.*, No. clxiv., p. 114).

The equations to EL, FM (perpendicular to DE) are

$$y - R\sin 2C\sin A = \cot A\,(x - R\sin 2C\cos A),$$
$$y - R\sin 2B\sin A = -\cot A\,(x + R\sin 2B\cos A)\,;$$

whence the y coordinate (*i.e.*, the a of trilinear coordinates) of the orthocentre J of DEF is

$$-R\cos 2A\cos(B-C)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

It is obvious that J lies on the line (2).

If O be the circum-centre of ABC, it follows, from (1) and (3), that ω is the mid-point of OJ.

Let α be the projection of E on BC, then

$$D\alpha = R\sin 2C\cos A,$$

and we find

$$(\omega\alpha)^2 = \frac{R^2}{4}(1+\cos 2A\cos 2B+\cos 2B\cos 2C+\cos 2C\cos 2A)$$
$$= R^2(\sin^2 A\sin^2 B\sin^2 C+\cos^2 A\cos^2 B\cos^2 C),$$

which shows that ωa is equal to the radius of Mr. H. M. Taylor's circle. (*Messenger of Mathematics*, Vol. xi., p. 177; *Proceedings of London Mathematical Society*, p. 138, *supra*. The circle is also obtained in Neuberg's *Sur le centre des médianes antiparallèles*, §16, where reference is also made to Catalan, *Théorèmes et Problèmes*, 6th edition, p. 133, and the *Nouvelle Correspondance*, T. vi. p. 183). This result was obtained originally in the following manner :—

Assume $\mu' = \cos A \cos B \cos C$;

now $\cos 2A = 1 - a^2/(2R^2)$, ... = ..., ... = ...;

hence, using notation employed in the "T. R. circle" article, we have

$$\kappa = a^2+b^2+c^2 = 8R^2(1+\mu'),$$

$$\lambda^2 = a^2b^2+b^2c^2+c^2a^2 = 4\Delta^2+\kappa^2/4,$$

and $8[\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C]$

$$= (1-\cos 2A)(1-\cos 2B)(1-\cos 2C) + (1+\cos 2A)(1+\cos 2B)(1+\cos 2C)$$

$$= 2[1+\cos 2A \cos 2B+\cos 2B \cos 2C+\cos 2C \cos 2A]$$

$$= 2[\Delta^2/(4R^4)+\mu'^2].$$

Now take $t = 1/(4R^2)$ in the expression $t^2\lambda^2 - t\kappa + 1$ [see "A Group of Circles," *Quarterly Journal*, Vol. xx., No. 77], and we have this value of t satisfies equation (viii.),

$$\sigma^2 = \frac{\lambda^2}{16R^4} - \frac{\kappa}{4R^2} + 1 = \frac{\Delta^2}{4R^4} + (1+\mu')^2 - 2(1+\mu') + 1$$

$$= \frac{\Delta^2}{4R^4} + \mu'^2 = 4(\sin^2 A \sin^2 B \sin^2 C + \cos^2 A \cos^2 B \cos^2 C],$$

which shows that Mr. Taylor's circle is a particular circle of the "group," as is obvious when we have obtained the result. The equation, then, to the circle is formed by giving t the above value in equations (ix.) or (x.) of "A Group of Circles."

We may note here that, if D'', E'', F'' are the mid-points of AP, BP, CP, then the nine-point centre is the mid-point of the line joining K', K'', the Symmedian points of $D'E'F'$, $D''E''F''$.

Mr. C. W. Merrifield, born at Brighton in 1827, died at Hove on January 1st, 1884, in his 57th year. Though called to the Bar, he never practised; but having in 1847 received from the then Lord Lansdowne an appointment in the Education Department of the Privy Council Office, he was soon promoted to the office of an examiner, the duties of which post he discharged, with marked attention and success, to within a few months of his death. He was an

early member of the Royal Institute of Naval Architects, and for many years acted as Honorary Secretary until his retirement from that post in 1875. In 1863 he had been elected a Fellow of the Royal Society. When the Royal School of Naval Architecture and Marine Engineering at South Kensington was established in 1867, Mr. Merrifield, at the request of the authorities, accepted the office of Vice-President. In consequence of the death of Mr. Purkiss, who was to have been the Principal, Mr. Merrifield was appointed to this latter post. Subsequently, on the transfer of the Institution to Greenwich, he resumed his office of examiner in the Education Department.

He was a frequent attendant at the meetings of the British Association, and in 1875 he was Vice-President of its section of Mechanical Science, and in the following year he was President of the same section.

Mr. Merrifield took very great interest in, and was very closely connected with, the Association for the Improvement of Geometrical Teaching, from its foundation, and was at the time of his decease one of its Vice-Presidents. Not to dwell longer on other incidents of his very busy and useful career, we now state what more immediately affects this Society. He was elected a member, March 19th, 1866; was elected a member of our Council, November 10th, 1870, and continued to be a member thereof until November, 1883, having resigned (December 14th, 1882), on the ground of ill-health, the office of Treasurer, which he had held from November, 1880. He was Vice-President in the Sessions 1876–77, 1877–78, and President in the Sessions 1878–79, 1879–80.

Mr. Merrifield was a most regular attendant both at Council and General Meetings, and took the warmest interest in the advancement of the Society. "His acquaintance with mathematical arithmetic, methods of interpolation, and tabular work in general, was very wide and complete."* He was also well versed in Greek and Latin, as well as in the classic authors in French and Italian, both of which languages he wrote well and spoke fluently—witness the evidence, as regards Italian, of the editor of the *Rivista Marittima*, in a foot-note to a paper "On Sea Waves," contributed by Mr. Merrifield to that journal. Mr. Merrifield edited many of the works in the Text Books of Science, published by Messrs. Longman, and wrote a treatise on "Technical Arithmetic and Mensuration" (1870) for that series. His contributions to mathematical journals were many, but not of any

* We are very greatly indebted to obituary notices in the *Sussex Daily News* of January 9th, 1884, and in *Nature*, Vol. xxix., p. 270, January 17th, 1884. From the notice in the *Proceedings of the Royal Society*, Vol. xxxvi., No. 231, we learn that the account in *Nature* was written by Mr. J. W. L. Glaisher, F.R.S.

great length. We give here a list of the most interesting papers and articles.

We begin with the papers in our own *Proceedings* :—

"On a Geometrical Proposition, indicating that the Property of the Radical Axis was probably discovered by the Arabs," *Proc.*, Vol. ii., pp. 175-77.

"Property of Conical and Cylindrical Surfaces," Vol. iii., pp. 222, 223, *Mess. of Math.*, 1871, pp. 81-88.

"Determination of the Form of the Dome of Uniform Stress," Vol. v., pp. 113-19.

"Note on a Numerical Theorem connected with the Cubical Division of Space," Vol. xi., pp. 157, 158.

"Considerations respecting the Translation of Series of Observations into Continuous Formulæ," Vol. xii., pp. 4-14. (Presidential Address.)

"On Ptolemy's Theorem," Vol. xii., pp. 142, 214.

"Note on the Limit to the Number of Different Proper Fractions whose Denominators are less than x, when x is large," Vol. xiii., pp. 240, 241.

Oxford, Cambridge, and Dublin Messenger of Mathematics :—

"An Approximation to the Value of π," Vol. iii., p. 105 ; 1866.

"Stirling's Series for Interpolation by means of Symmetric Differences," Vol. iv., pp. 110-113.

"Resistance as the Cube of the Velocity," Vol. v., pp. 5-12.

"A Verification of the Solution of a Particular System of Simultaneous Equations," Vol. v., pp. 206, 207.

Messenger of Mathematics :—

Vol. i.—"On Certain Families of Surfaces," pp. 81-87.

Vol. iii.—"Cartesian Ovals regarded as Projected Intersections of Surfaces of the Second Degree," pp. 141, 142.

Vol. iv.—"A Theorem in Transversals," p. 21.

"Cartesian Ovals considered as Conic Intersections," p. 148.

"Elementary Proof of the Formula which expresses the Absence of the Molecular Rotation in Fluid Motion," pp. 105-107.

Vol. v.—"Anharmonic Ratio of a Pencil of Five Lines in Space," pp. 94-98.

"Isotropy in Homogeneous Solids," pp. 113-120.

"On the Geometry of the Elliptic Equation," *Phil. Mag.* (1858), xvi., pp. 198-202.

"On the Comparison of Hyperbolic Arcs," *Phil. Trans.* (1859), pp. 171-75.

"On a new Method of Approximation applicable to Elliptic and Ultra-Elliptic Functions," *Roy. Soc. Proc.*, x., pp. 474, 475 ; *Phil. Trans.* (1860), pp. 223-28.

(Second Memoir.) *Roy. Soc. Proc.*, xii., pp. 57, 58 ; *Phil. Trans.* (1862), pp. 417-35.

"On Approximation to the Integrals of Irrational Functions by means of Rational Substitutions," *Phil. Mag.*, xx. (1860), pp. 446-48.

"Notes on the Hexahedron Inscribed in a Sphere," *Phil. Mag.*, xxii. (1861), pp. 382-84.

"On the Comparative Value of Simpson's Two Rules, and on Dr. Woolley's Rule," *Naval Architects' Trans.*, vi., 1865, pp. 40-48. (Six other papers were contributed to these *Transactions* in the years 1865, 1866, 1867.)

"Example of the Application of a Graphical Method to the Problem of Rectilinear Motion in a Homogeneous resisting Medium," *Phil. Mag.*, xxxv. (1868), pp. 420-23.

"On the Law of the Resistance of the Air to Rifled Projectiles," *Phil. Trans.*, clviii. (1868), pp. 443-46.

(Similar papers contributed to other journals.)

"On a New Method of Calculating the Statical Stability of a Ship," *Roy. Soc. Proc.*, xv. (1867), pp. 332-5.

[His views on this and conjugate subjects will be found also embodied in *Brit. Assoc. Reports* for 1869, pp. 10-47; 1872, pp. 246, 247; *Roy. Sch. Naval Architecture Annual*, No. 1 (1871), pp. 51-56.]

"Deep Sea Waves"—Naval Science, ii. (1873), pp. 85-100.

"A Summary of the Theory of the Oscillating Sea Wave," *Roy. Sch. Naval Arch. Annual*, No. 3 (1873), pp. 26-36.

[We are indebted for many of these to the Royal Society Catalogue, which extends only to the year 1879.]

"A Catalogue of a Collection of Models of Ruled Surfaces constructed by M. Fabre de Lagrange, with Appendix" (1872).

"Miscellaneous Memoirs on Pure Mathematics" (from the *Assurance Magazine*, 1861), privately printed.

"Review of Cayley's Elliptic Functions," *Nature*, Vol. xv., pp. 252, 253 (January, 1877).

"Report on the Present State of Knowledge of the Application of Quadratures and Interpolation to Actual Data" (read at British Association, Swansea Meeting, 1880, 58 pp.).

Mr. Merrifield was also a frequent contributor to the Mathematical Questions in the *Educational Times*.

Dr. Isaac Todhunter, M.A., was born at Rye, on the 22nd November, 1820, and died at Cambridge, March 1st, 1884. For a very full "In Memoriam" sketch we refer to the numbers of the *Cambridge Review*, for March 5th, 12th, 19th, written by the skilled pen of Professor Mayor.* "We of his year said of him that he might have been Senior Wrangler when a freshman; he had strong men to compete with—Barry and Mackenzie,—but it was never doubtful that the first place in the Tripos and the first Smith's Prize would fall to him." He was Senior Wrangler in 1848 (having gained the Gold Medal at his M.A. London Examination in 1847), and in due course of time got his Fellowship at St. John's College, and became Principal Lecturer; he was Senior Moderator in 1865 and Senior Examiner in the year following, but the work he considered to be so onerous that he never undertook either office again. He became a Fellow of the Royal Society in 1862, and served on the Council during the years 1871–73. After his marriage he was elected an Honorary Fellow of his College, and, "when the University established its new Doctor of Science degree, he was one of those whose application was granted within the first few months." During our undergraduate course (1851–55) we attended his college lectures, with but little profit, but that doubtless

* The *Memoir* has been reprinted and issued in book form, with additional particulars (Cambridge: Macmillan & Bowes, 60 pp.). It has been stated that it is Prof. Mayor's intention to write a fuller Memoir.

was owing to our own crassness.* We always found him ready to give advice as to plans of study, but we could never see how the work to be done could be got into an ordinary day. The acquaintance then begun continued up to the time of his death, and it affords us a melancholy pleasure, yet a very real one, that we renewed a personal intercourse, intermitted for so many years, during a brief sea-side sojourn at Hunstanton, Norfolk, in September, 1883. Our epistolary intercourse had been well maintained in the interim, and we had ever found him to be a most prompt and courteous correspondent. Our letters were mainly concerned with the contents of his numerous text-books. To the first edition of his *Plane Trigonometry*, we furnished upwards of 150 corrections or suggestions. In the later editions of these works, we hazard the statement that it would be almost as difficult to find an error as to find the proverbial needle in a bundle of hay. It hardly seems worth while to expend so much pains upon such books; he himself wrote, "There are fluctuations of taste and fashion even in the serene realms of abstract science,"† and in another place he remarks of an older text-book that it "sooner passed into that obscurity which is the fate of all academical text-books."‡ Whatever may be the fate of these minor works hereafter, there can be no question that they have hitherto enjoyed a high measure of success, and they appear to us to be well adapted for beginners and self-taught students, for nothing is assumed but what the learner should know, and the work is put in such a form that such a one can see how the results in many cases are got. His fame as a writer, however, will indubitably rest upon his charming histories, those "great historical treatises, so suggestive of research and so full of its spirit,"§ which have caused a distinguished pupil of his own, even when taking him to task for his non-appreciation of "experiments" in teaching, to style him "one of the most erudite and voluminous of British mathematicians."‖ In his essay on "Elementary Geometry," Dr. Todhunter speaks of his "taste for the history

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* In those early days he came out of his sanctum, placed a paper of questions on the table, and then retired. He was accessible to inquiries, but our own recollection of him is that there was nothing whatever said in the way of lecturing. Curiosity took us once or twice to the door of the said sanctum, where we could see the long rows of books, all attired, if memory serves us, in the same monotonous "grocers' paper" covers. It was an eccentricity.

† "W. Whewell. An account of his writings, with selections from his . . . . correspondence," by I. Todhunter (1876), Vol. i., p. 407.

‡ The same, Vol. i., p. 24.

§ Prof. H. J. S. Smith in his Presidential Address, *Proc.*, Vol. viii., p. 27.

‖ Prof. P. G. Tait in *Nature*, Vol. ix., p. 323. One is at times disposed to think that Todhunter is "poking fun" at the clergy in this passage—as he does elsewhere —but the whole paragraph does not warrant this view.

of mathematics,"* a taste doubtless not diminished by his training under De Morgan.

We take it that there was no deep vein of originality, *quoad* mathematics, in Dr. Todhunter; but his taste referred to above, his powers of patient investigation and of weighing evidence, admirably fitted him for the task he, happily for students, undertook, and we feel assured that his name will always be honoured for the good work he has done in this field of research. A distinguished mathematician, whose personality is thinly veiled under initials, remarks :—"As we read one of them, it seems as if a new light were thrown on the subject. . . We see here sketched out before us the gradual growth of those modern methods which we now find so ready to our hands."†

In 1849, Dr. Todhunter gained the Burney Prize for the best essay on "The Doctrine of a Divine Providence is inseparable from the Belief in the existence of an absolutely perfect Creator," and in 1871 he was awarded the Adams Prize for an essay entitled "Researches in the Calculus of Variations." It will be thus seen that his range of reading was widely diversified; in addition, Prof. Mayor states, to being a sound Latin and Greek scholar, he had studied Hebrew, Arabic, Persian, and Sanscrit, and, besides an acquaintance with most European languages, had learnt enough of Russian to master a mathematical treatise.

Dr. Todhunter was not, as we should have expected, an original member of the Society, but he was elected at a very early date, viz., on June 18, 1865. He never held any position on the Council, nor did he ever act as a referee, always pleading to be exempted from such offices on the ground of his being so very much occupied in other ways. These engagements were, in part, no doubt, the cause of his contributing so few papers to scientific journals. We give a list of those we have met with :—

"On the Method of Least Squares," *Phil. Mag.*, xxx., 1865, and *Camb. Phil. Soc. Trans.*, 1865 (Vol. xi., Pt. 2).

"Note on a Paper by Balfour Stewart on a Proposition in the Theory of Numbers" (1865), *Edin. Roy. Soc. Proc.*, v., 1866, pp. 517, 518.

"Property of a Regular Polyhedron," *Oxford, Camb., and Dub. Mess. of Math.*, Vol. iii. (1866), p. 142.

"On a Problem in the Calculus of Variations," *Phil. Mag.*, xxxi., 425-27; xxxii., 199-205; xlii., 440, 441.

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* "The Conflict of Studies and other Essays" (1873), p. 151.

† Obituary Notice, *Proc. of the Royal Society*, Vol. xxxvii., No. 233. (Signed E. J. R.) The present writer reviewed, at some length, the work on "Attraction" in *Nature*, Vol. ix., pp. 378-380, 399-401. In a letter of date March 28, 1874, (now before us) Dr. Todhunter states,—"I have read with great interest your valuable remarks on my History. . . . Such books as this History are not likely to be reprinted; but I hope some day to publish notes and corrections for all my Histories in one small volume." May we hope that such a MS. has been left behind in a fit state for publication? We have seen no mention of it.

"On the Method of Demonstrating some Propositions in Dynamics," *Camb. Phil. Soc. Proc.*, ii., 1866, pp. 16-18.

"On Jacobi's Theorem respecting the Relative Equilibrium of a Revolving Ellipsoid of Fluid, and on Ivory's discussion of the Theorem," *Proc. of Roy. Soc.*, No. 123 (1870), pp. 42-56.

"On Prop. 38 of the Third Book of Newton's Principia," *Roy. Astr. Soc. Monthly Notices*, xxxii. (1872), pp. 234-36.

"Note relating to the Attraction of Spheroids," *Proc. of Roy. Soc.*, No. 137 (1872), pp. 507-513.

"On the Arc of the Meridian measured in South Africa," *Roy. Astr. Soc. Monthly Notices*, xxxiii., pp. 27-34.

"On the Arc of the Meridian measured in Lapland," *Camb. Phil. Soc. Trans.*, Vol. xii., Pt. i., pp. 1-26.

"On the Equation which Determines the Form of the Strata in Legendre's and Laplace's Theory of the Figure of the Earth," *Camb. Phil. Soc. Trans.*, Vol. xii., pp. 301-18.

"Note on the History of Certain Formulæ in Spherical Trigonometry," *Phil. Mag.*, xlv. (Feb. 1873), pp. 98-100. [cf. *Proc.*, Vol. iii., pp. 319, 320.]

"Note on an Erroneous Extension of Jacobi's Theorem," *Proc. of Roy. Soc.*, No. 141 (1873), pp. 119-121.

Dr. Todhunter, at the time we saw him, told us that he had nearly finished a treatise on "Elasticity," on the model of his previous Histories. "The time and labour he spent over this work injured his eyesight, and probably led to his final illness." Prof. Cayley has reported favourably upon the condition of the MS., and we are glad to know that the seeing of the work through the press has been entrusted to such a competent editor as Prof. Karl Pearson. We fear, however, that the task will not be a light one, as there are several *lacunæ* which have to be filled up. "Another result of the labours of his latter years is a treatise on Arithmetic." But we have not heard whether this was advanced enough for publication.

Professor Richard Charles Rowe, M.A., (he had only just vacated his Professorship,) died suddenly at Farnham Royal, Bucks, on September 21st, 1884. He was educated at the Bristol Grammar School; thence, having passed all his examinations for the degrees of B.A., M.A., and B.Sc., at the London University with distinction, in 1873 he entered at Trinity College, Cambridge, graduating B.A. in 1877, as Third Wrangler. He was subsequently bracketed second Smith's Prizeman. He was elected Fellow of his College in October, 1878. After a short interval, he was elected in 1880 Professor of Pure Mathematics at University College, London. He was an Examiner for the Mathematical Tripos in 1881, and also held the same office at the time of his lamented death. In him "Trinity College loses a mathematician of great ability; his friends, a man of childlike purity and singular attractiveness; and Cambridge lovers of music, a pianoforte-player and exponent of the great masters such as we cannot

reasonably hope to meet again."* The same authority remarks that "Pure Mathematics in Cambridge loses by his death one of her most faithful admirers, whose influence for good would have made itself more and more felt in the University with each succeeding year."

Professor Rowe joined our ranks November 13, 1879, and served on the Council in the two Sessions 1881–82, 1883–84.†

His published writings are two only:—"Note on a Geometrical Theorem" (written in conjunction with Mr. H. M. Taylor), *Proceedings*, Vol. xiii., pp. 102—106; and a paper published in the *Phil. Trans.* (for 1881, pp. 713—750). "He was elected a Fellow of Trinity . . . . on a brilliant examination, including the production of a dissertation in which Abel's theorem—one of the most general in the whole range of mathematics, and well beyond the region of Cambridge studies—was treated in a masterly manner, and its mode of statement greatly improved and simplified. . . . It is by this work alone that Rowe will be known by future generations of mathematicians."

Professor Rowe undertook to edit "The Common-sense of the Exact Sciences," which had been left in an unfinished state by the late Professor Clifford, and had, at the time of his death, advanced some way with the work. The task of getting the book out, now somewhat complicated and requiring considerable delicacy of treatment, has been offered to and accepted by Professor Karl Pearson, the present occupier of Clifford's post at University College.

The Rev. Richard Townsend, M.A., died on the 17th October, 1884, in his 64th year, at his residence, Upper Leeson Street, Dublin, "after a protracted and complicated illness, of which heart-disease was the most serious element." We have not been able to meet with any account of his mathematical work, but a very interesting obituary sketch, by Prof. Mahaffy, will be found in the *Athenæum* (Vol. July to Dec., 1884, Oct. 25, p. 532). Mr. Townsend is reported to have obtained his Fellowship in 1845, upon distinguished answering, and we know that he was one of the most accomplished and successful tutors in Trinity College. During a residence of five years in Ireland, we sent up many young pupils to the College, and all were unanimous in singing his praises, so that we can add our testimony to the truth of the statement that "his unaffected manner and generous kindness made him a great favourite with every student, and, if the rules of the College had allowed, they would all have had him for their tutor."

In 1866 he became a Fellow of the Royal Society, and in 1870 was appointed Professor of Natural Philosophy in the University. He is perhaps best known by his "Chapters on the Modern Geometry of the

* Mr. Sedley Taylor in the *Cambridge Review* of October 15th, 1884.

† He was elected by the Council, April 12th, 1883, to fill the vacancy caused by Prof. H. Smith's death.

Point, Line, and Circle; being the substance of lectures delivered in the University of Dublin to the candidates for honors of the first year in Arts" (1863);* a work in two volumes octavo.

Prof. Townsend was elected a member, April 16, 1866. His great distance from London stood in the way of his being a member of the Council; he, however, often rendered good service by acting as a referee whenever called upon by the Council to do so.

We understand that a work on "Attractions," upon which Mr. Townsend had been for some time engaged, is in a sufficiently finished state to be published. His first recorded paper is "On Principal Axes of a Body, their Moments of Inertia, and Distribution in Space," *Camb. and Dublin Math. Jour.*, i. (1846), pp. 209–227; ii., pp. 19–42, 140–171, 241–251. He contributed six other papers to this journal down to Vol. viii. (1853). To its successor, the *Quar. Jour. of Math.*, down to the year 1873 (*Roy. Soc. Catalogue*), he contributed ten papers. These are upon the subjects to which he devoted most of his efforts, viz., Geometry, Dynamics, and Attraction, and are familiar to our members. The only paper upon which M. Chasles (in his *Rapport sur les Progrès*, &c., p. 371, 1870) comments is that "On a Property of the Director Spheres of a System of Quadrics touching a Common System of Planes," *Quar. Jour. of Math.*, T. viii., No. 29 (1866), pp. 10–14.

One paper only appears in our *Proceedings*:—"Solution, by the method of ordinary Homographic Division, of the Problem, 'To Inscribe in a given Ruled Quadric a Polygon of any given order, whose sides shall pass in any prescribed order of sequence through an arbitrary system of given points in space,'" Vol. ii., pp. 21–24.

The notelets which follow are from the *Messenger of Mathematics*:—

Vol. i.—"On a Property in the Theory of Confocal Quadrics," p. 49; "On an Analogue in the Theory of Quadrics to a known Property in the Theory of Conics," p. 70.

Vol. ii.—"On a Property of the Wave Surface," pp. 28, 29; "On a Property in the Theory of Confocal Conics and its Analogue in the Theory of Confocal Quadrics," pp. 33-35.

Prof. Townsend has been for many years a most assiduous contributor to the columns of the *Educational Times*, both as proposer of original questions and as solver of questions proposed by others. These contributions were marked by the elegance of style for which their author has been so long known. He used to excuse himself from sending communications on the score of his time being so taken up with lectures and other work connected with his tutorial office.

R. T.

* We ought hardly to tell it in Gath, but in the *British Museum Catalogue* we can only find the title as above, with the words attached "*in progress.*" This makes it appear to the uninitiated as if the work were appearing in parts at intervals, more or less definite.

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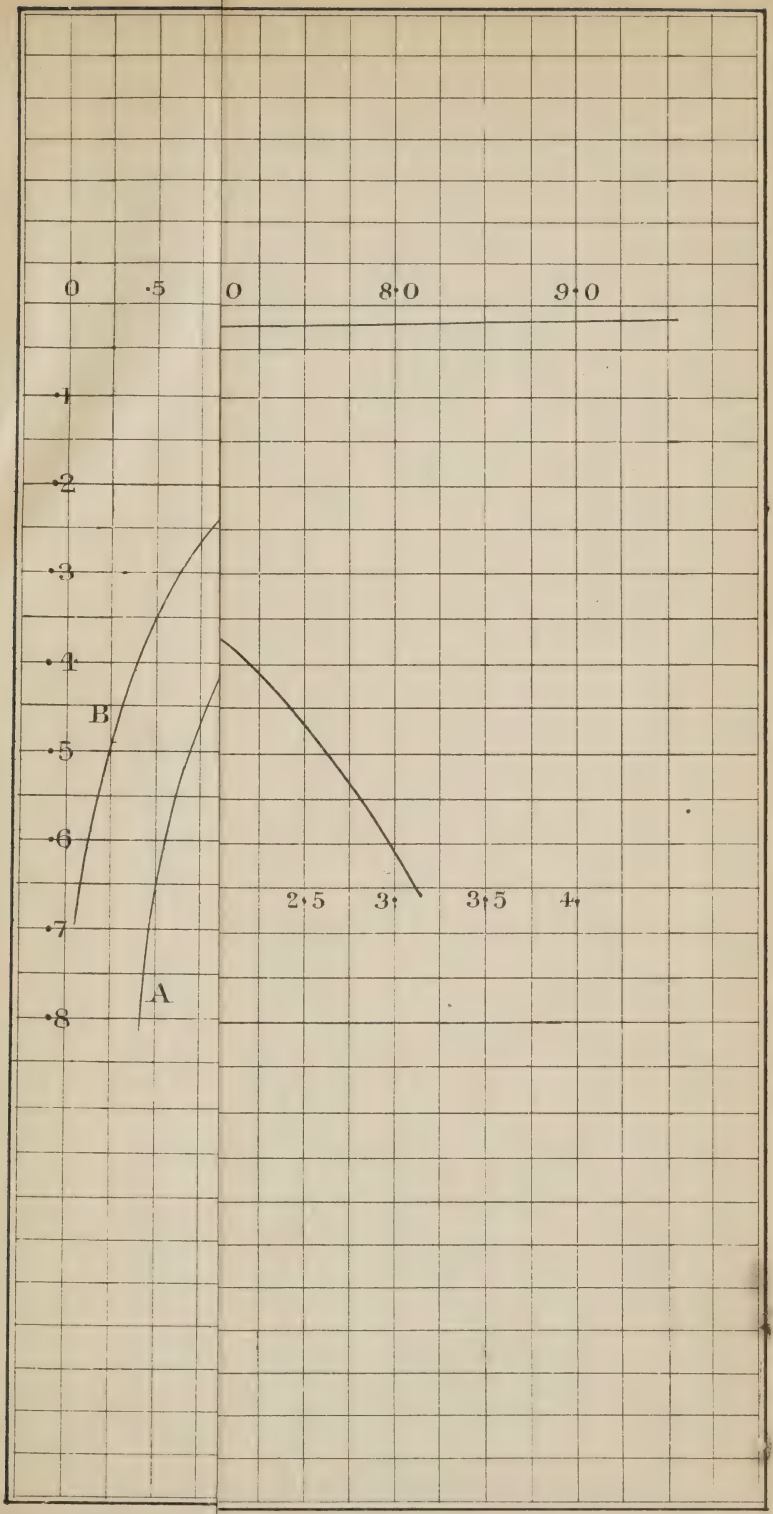
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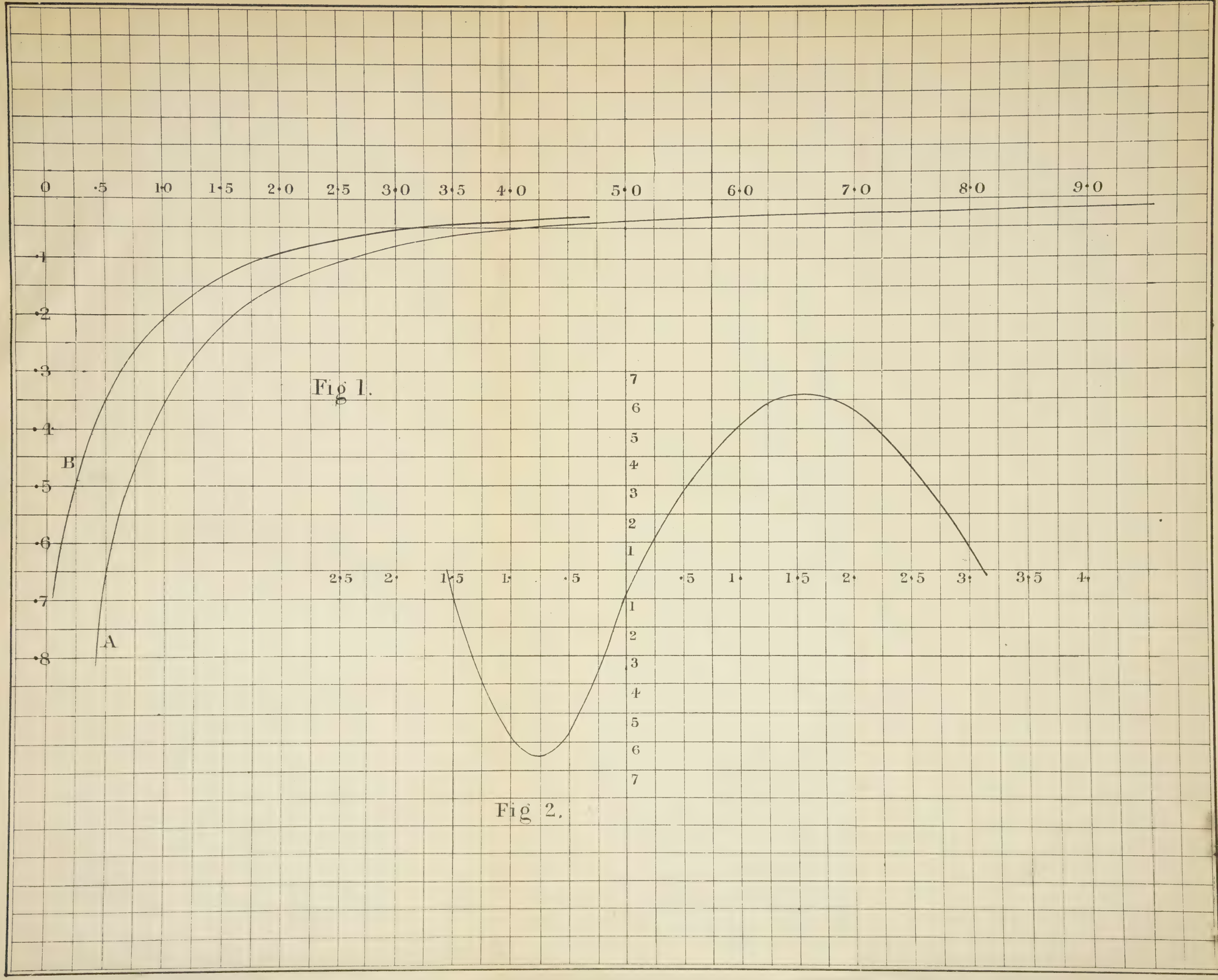
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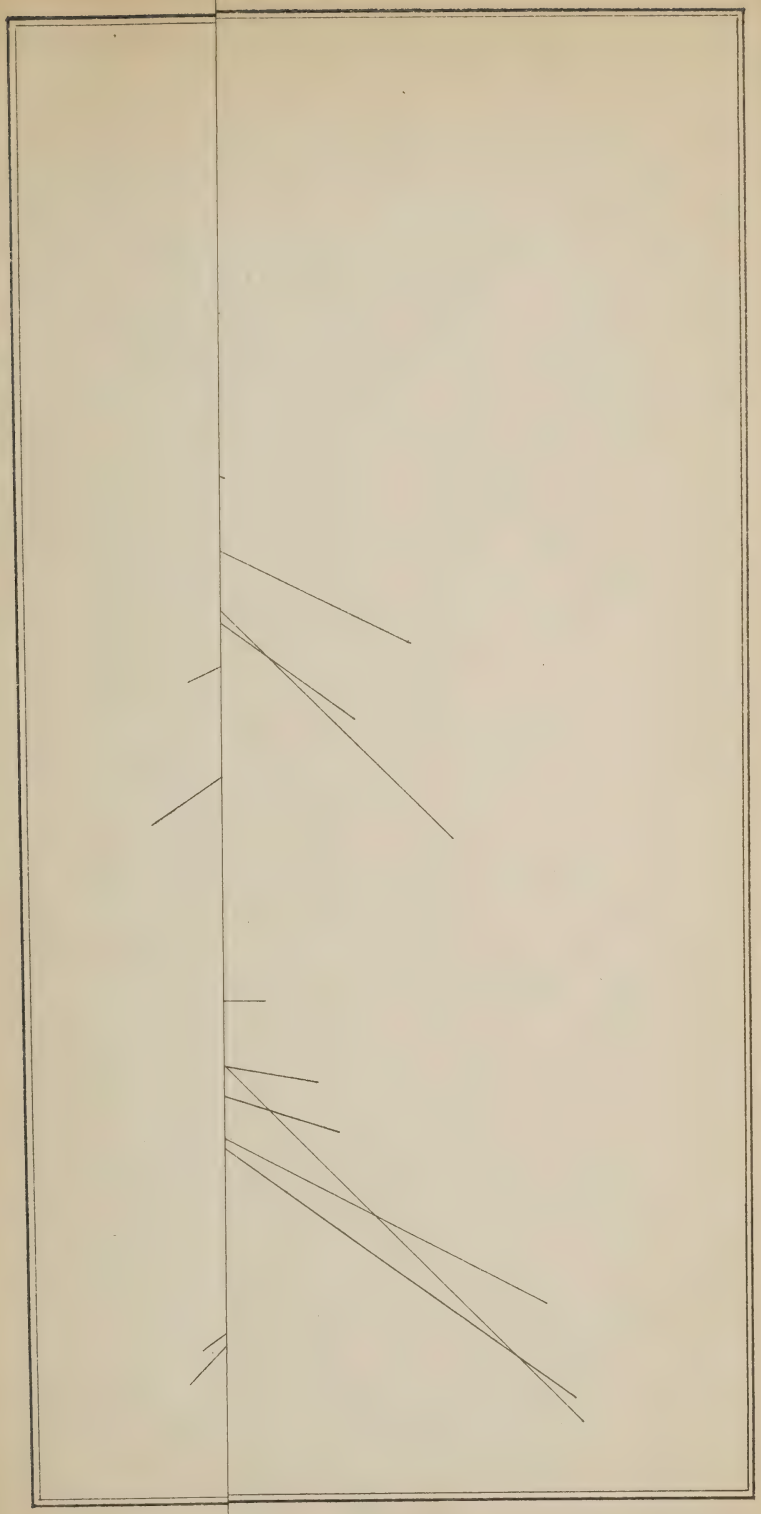

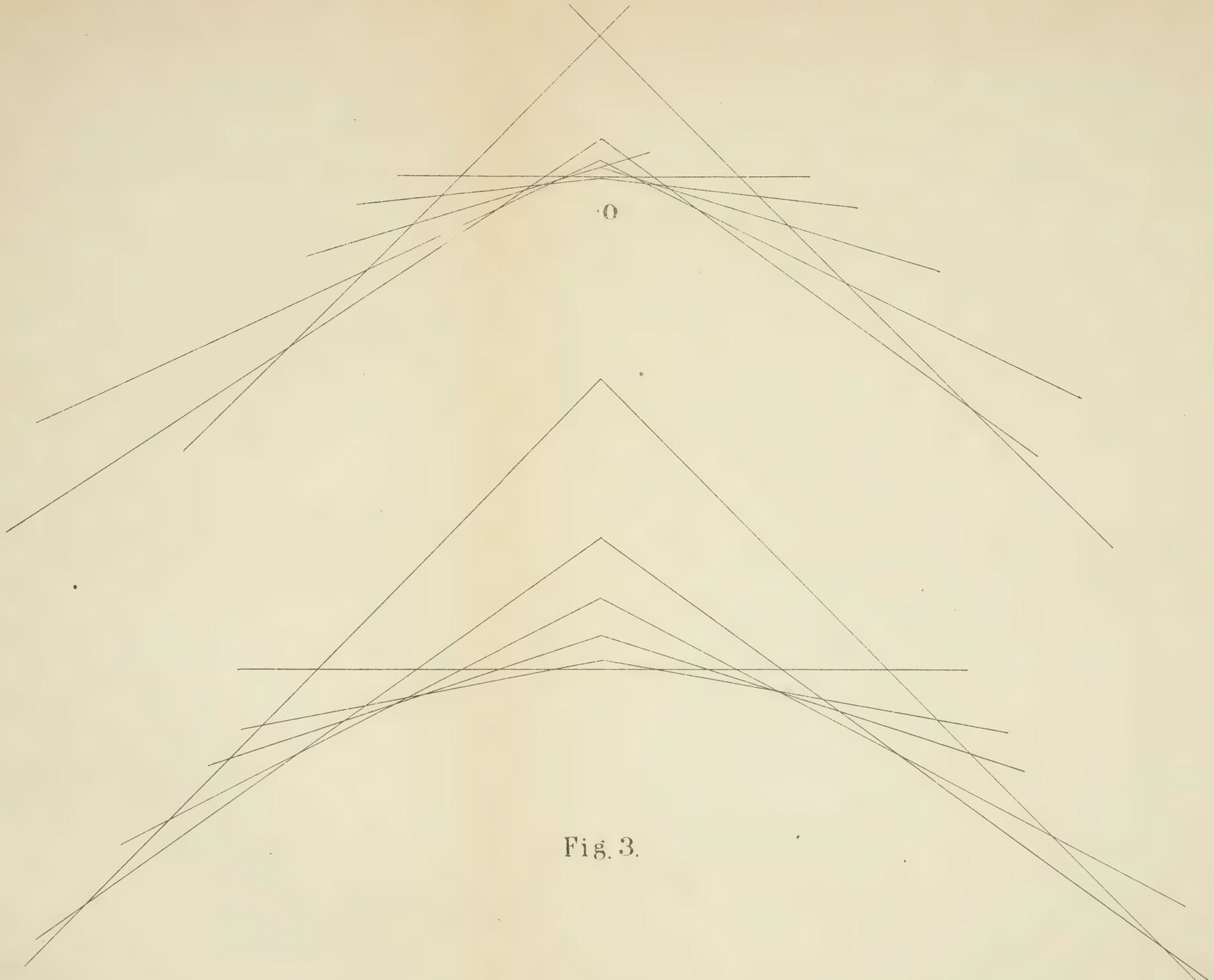

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